Portfolios

A portfolio is a collection of assets.
- The **weights** fundamentally define a portfolio.
- The **weights** are the proportion invested in each asset

Portfolios can contain many assets.
- A portfolio can be completely comprised of risky assets.
- If you own only one asset do you still have portfolio?
- Yes, a pretty simple one.
Diversification

The principle of diversification is fundamental to finance. Portfolios allow an investor to become diversified.

Which bet do you prefer?

- Flip a fair coin once; if it is heads I will give a $1000, you get nothing if it is tails.
- Flip a fair coin 1000 times; each time you flip heads I will give you $1.00. I will give you nothing for tails.

Most people prefer the second option; most of the risk is diversified away through multiple flips.

You own Dell

Suppose you own only one security, Dell Computer Corp. What sources of risk do you face?

Systematic risk

- What is systematic risk?
- What are some other names for systematic risk?
- Can you think of some concrete examples?

Idiosyncratic risk

- What is idiosyncratic risk?
- What are some other names for idiosyncratic risk?
- Can you think of some concrete examples?
Diversification

Half in Dell and half in PepsiCo

Suppose instead you put half your money in Dell and half in PepsiCo.

What happens to your portfolio’s risk?

- Why is idiosyncratic risk reduced?
- Aren’t you subject to more idiosyncratic risk? Bad things can happen to both Dell and Pepsi?
- Why is systematic risk unaffected?

Adding Securities: Systematic and Idiosyncratic Risk

A: Firm-specific risk only

B: Market and unique risk

Unique risk

Market risk
Adding Stocks to An Equal Weight Portfolio

Diversification

Defining a diversified portfolio
- Are all portfolios with lots of securities diversified?
- Do other conditions need to be met to call a portfolio diversified?
Statistical Detour: Covariance

**What is covariance?**
Covariance is a measure of how much two variables move together.

**Consider a few examples (source: Welch)**

<table>
<thead>
<tr>
<th>Negatively Covary</th>
<th>Zero Covariation</th>
<th>Positively Covary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agility vs. Weight</td>
<td>IQ vs. Gender</td>
<td>Wealth vs. Longevity</td>
</tr>
<tr>
<td>Wealth vs. Disease</td>
<td>Age vs. Blood Type</td>
<td>Arm Strength vs. QB Quality</td>
</tr>
</tbody>
</table>

**Computing Covariance**
If $x$ and $y$ are random variables that take on the values $x_i$ and $y_i$ with probability $p_i$,

$$
cov(x, y) = \sum_{i=1}^{n} p_i [x_i - E(x)] [y_i - E(y)]
$$

If $A$ and $B$ are securities, then the covariance between the return on $A$ and the return on $B$ is written as the following:

$$
cov(r_a, r_b) = \sum_{i=1}^{n} p_i [r_{ia} - E(r_a)] [r_{ib} - E(r_b)]
$$
Spam and Donuts

You want to invest in Hormel and Krispy Kreme

What is the covariance between their returns (cov(r_{hrl}, r_{kk}))?

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Probability</th>
<th>r_{hrl}</th>
<th>r_{kk}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good</td>
<td>0.30</td>
<td>0.12</td>
<td>0.20</td>
</tr>
<tr>
<td>Normal</td>
<td>0.40</td>
<td>0.08</td>
<td>0.10</td>
</tr>
<tr>
<td>Bad</td>
<td>0.30</td>
<td>0.08</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Step 1: Compute the expected returns of Hormel and Krispy Kreme

\[ E(r_{hrl}) = 0.3(0.12) + 0.4(0.08) + 0.3(0.08) = 9.2\% \]
\[ E(r_{kk}) = 0.3(0.20) + 0.4(0.10) + 0.3(0.00) = 10\% \]

Step 2: Use the covariance formula

\[
\text{cov}(r_{hrl}, r_{kk}) = p_g[r_{g,hrl} - E(r_{hrl})][r_{g,kk} - E(r_{kk})] + \\
p_n[r_{n,hrl} - E(r_{hrl})][r_{n,kk} - E(r_{kk})] + \\
p_b[r_{b,hrl} - E(r_{hrl})][r_{b,kk} - E(r_{kk})]
\]

\[
= 0.3(0.12 - 0.092)(0.20 - 0.10) + \\
0.4(0.08 - 0.092)(0.10 - 0.10) + \\
0.3(0.08 - 0.092)(0.00 - 0.10)
\]

\[
= 0.0012
\]

What does 0.0012 tell us?

Does cov(r_{hrl}, r_{kk}) = 0.0012 mean a strong relation or a weak one?
Correlation

Computing Correlation

If $x$ and $y$ are random variables, then the correlation between $x$ and $y$ ($\rho(x, y)$) is the following:

$$
\rho(x, y) = \frac{\text{cov}(x, y)}{\sigma(x)\sigma(y)}
$$

If $A$ and $B$ are securities, then the covariance between the return on $A$ and the return on $B$ is written as the following:

$$
\rho(r_a, r_b) = \frac{\text{cov}(r_a, r_b)}{\sigma(r_a)\sigma(r_b)}
$$

Correlation: -1 to 1

Correlation ranges from -1 to 1

- -1, indicates perfect negative correlation.
- 0, indicates that the two variables are unrelated.
- +1, indicates perfect positive correlation.
Correlation: Spam and Donuts

What is $\rho(r_{hl}, r_{kk})$?

$$
\sigma(a) = \sqrt{0.3(0.12 - 0.092)^2 + 0.4(0.8 - 0.092)^2 + 0.3(0.08 - 0.092)^2} = 1.8\% \\
\sigma(b) = \sqrt{0.3(0.20 - 0.10)^2 + 0.4(0.10 - 0.10)^2 + 0.3(0.00 - 0.10)^2} = 7.7\%
$$

$$
\rho(r_{hl}, r_{kk}) = \frac{\text{cov}(r_{hl}, r_{kk})}{\sigma(r_{hl})\sigma(r_{kk})} \\
= \frac{0.0012}{(0.018)(0.077)} = 0.87
$$

Estimating Covariance

Estimation

Usually we have to estimate covariances and correlations using past data just like variances.

Estimating Covariance

if $r_{i1}, r_{i2}, r_{i3}, \ldots r_{iT}$ are one-period returns for security $i$, $r_{j1}, r_{j2}, r_{j3}, \ldots r_{jT}$ are one-period returns for security $j$, then the sample or estimated covariance is the following:

$$
\hat{\text{cov}}(r_i, r_j) = \frac{1}{T-1} \sum_{t=1}^{T} (r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)
$$

where $\bar{r}_i$ and $\bar{r}_j$ are the sample means of returns for security $i$ and $j$:

$$
\bar{r}_i = \frac{1}{T} \sum_{t=1}^{T} r_{it} \quad \text{and} \quad \bar{r}_j = \frac{1}{T} \sum_{t=1}^{T} r_{jt}.
$$
Estimating Correlation

The estimated or sample correlation coefficient

\[ \hat{\rho}(r_i, r_j) = \frac{\hat{\text{cov}}(r_i, r_j)}{\hat{\sigma}(r_i) \hat{\sigma}(r_j)} \]

where \( \hat{\sigma}(r_i) \) is the sample standard deviation of security \( i \) and \( \hat{\sigma}(r_j) \) is the sample standard deviation of security \( j \).

A Portfolio of Two Risky Assets

The return on a two-asset portfolio:

\[ r_p = w r_a + (1 - w) r_b \]

- \( r_a \) is the return and \( w \) is the weight on asset A.
- \( r_b \) is the return and \( 1 - w \) is the weight on asset B.
- Note: the weights of the portfolio sum to one.
A Portfolio of Two Risky Assets

Two Asset Portfolio: Expected return and standard deviation

If A and B are assets and $w$ is the portfolio weight on asset A, then the expected return of a portfolio composed of A and B is

$$E(r_p) = wE(r_a) + (1 - w)E(r_b),$$

the variance of the portfolio is

$$\sigma^2(r_p) = w^2\sigma^2(r_a) + (1 - w)^2\sigma^2(r_b) + 2w(1 - w)\text{cov}(r_a, r_b),$$

and the standard deviation of the portfolio is

$$\sigma(r_p) = \sqrt{\sigma^2(r_p)}.$$
Asset Allocation: Two Risky Assets

Example: A stock index and bond index

<table>
<thead>
<tr>
<th></th>
<th>$E(r)$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Index</td>
<td>12%</td>
<td>20%</td>
</tr>
<tr>
<td>Bond Index</td>
<td>4%</td>
<td>7%</td>
</tr>
</tbody>
</table>

$\rho(r_s, r_b) = -0.10$, $\text{cov}(r_s, r_b) = -0.0014$

What are the feasible allocations?

Try different weights: compute $E(r_p)$ and $\sigma(r_p)$ each time:

$$E(r_p) = wE(r_s) + (1 - w)E(r_b)$$

$$\sigma^2(r_p) = w^2\sigma^2(r_s) + (1 - w)^2\sigma^2(r_b) + 2w(1 - w)\text{cov}(r_s, r_b)$$
Asset Allocation: Two Risky Assets

Which portfolio should you choose?
In general it’s a matter of preference (for risk and expected return).

Can we rule out some portfolios?
- Are some portfolios just plain stupid to hold?
- Is it just plain stupid to hold just bonds (100% in the bond)?

Investment Opportunity Set: Changes in $\rho$

- $\rho = -1$
- $\rho = 0$
- $\rho = 1$
The Shape of the Investment Opportunity Set

The investment opportunity set

- The shape of the investment opportunity set is determined by the covariance between the assets.
- If $\rho$ is sufficiently low we can find a portfolio with lower variance than either of the assets.
- If two assets are perfectly negatively correlated you can diversify away all risk.

Adding a Riskfree Asset

Adding a risk free asset

Suppose that we also can invest in a riskfree asset with a return of 3%.

How does adding a riskfree asset change things?

- What does the investment opportunity set look like now?
- Does adding a riskfree rate increase the number of portfolios that a rational and risk averse investor should avoid?
- Is there now a single optimal risky portfolio?
Investment Opportunity Set

E(r) vs. \( \sigma(r) \)

CAL 1
CAL 2

Karl B. Diether  (Fisher College of Business)  Portfolio Theory: Two Risky Assets  27 / 33

Investment Opportunity Set

E(r) vs. \( \sigma(r) \)

CAL 1
CAL 2

Karl B. Diether  (Fisher College of Business)  Portfolio Theory: Two Risky Assets  28 / 33
Which CAL is the Best?

Clearly CAL 3 is the best of the three. For a given standard deviation it offers more expected return. Also, for a given expected return it has a lower variance. The risky portfolio that defines CAL 3 is better than the risky portfolios that define CAL 1 and CAL 2.

What is the optimal CAL?
It is CAL with the highest possible slope or the best expected return – standard deviation tradeoff.
The Optimal CAL and the Tangency Portfolio

The optimal CAL is the one with the highest slope (Sharpe ratio)

\[ \text{Slope} = \frac{E(r_{\text{risky}}) - r_f}{\sigma(r_{\text{risky}})} \]

Properties and implications

- The optimal CAL is the tangency line between the riskfree rate and the risky investment opportunity curve.
- The risky portfolio that the optimal CAL is tangent with is called the **tangency portfolio**.
- Investors will combine the tangency portfolio with the riskfree asset to form the overall portfolio that best suits their preferences for risk and expected return.
Looking Forward

**Mean variance analysis**
We will generalize our mean-variance analysis to portfolios with many risky assets.

**Models**
We will also introduce our first model based on mean-variance analysis (The CAPM).