On the theory of rubber friction

B.N.J. Persson *

Institut für Festkörperforschung, Forschungszentrum Jülich, D-52425, Jülich, Germany

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Abstract

Rubber exhibits unusual sliding friction. When rubber is slid on a hard, rough substrate, the surface asperities of the substrate exert oscillating forces on the rubber surface leading to energy "dissipation" via the internal friction of the rubber. I estimate this contribution to the friction force and compare the results with the experimental data of Grosch. Because of its low elastic modulus, the adhesion of the rubber to the substrate is very important. I show that the adhesion force will deform the rubber at the rubber–substrate interface, in such a manner that, at low sliding velocities, the rubber completely follows the short-wavelength surface roughness profile. This gives an additional contribution to the friction force, which is estimated and compared with experimental data. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The nature of the friction when rubber is slid on a hard substrate is a topic of considerable practical importance, e.g. for the construction of tires and wiper blades [1]. Rubber friction differs in many ways from the frictional properties of most other solids. The reason for this is the very low elastic modulus of rubber and the high internal friction exhibited by rubber in a wide frequency region.

The pioneer studies of Grosch [2] have shown that rubber friction, in many cases, is directly related to the internal friction of the rubber. Thus, experiments with rubber surfaces sliding on silicon carbide paper and glass surfaces give friction coefficients with the same temperature dependence as that of the complex elastic modulus E(ω) of rubber. This proves that the friction force is mainly a bulk property of the rubber.

The friction force between rubber and a rough (hard) surface has two contributions commonly described as the adhesion and hysteric components, respectively [1]. The hysteric component results from the internal friction of the rubber. During sliding, the asperities of the rough substrate exert oscillating forces on the rubber surface leading to cyclic deformations of the rubber and energy "dissipation" via the internal damping of the rubber. This contribution to the friction force will, therefore, have the same temperature dependence as that of the elastic modulus E(ω) (a bulk property). The adhesion component is important only for very clean rubber surfaces (see below).

In this paper we discuss both the adhesion and hysteric components of rubber friction. These topics have been considered before [1,3–6], but
the results presented below are new. First we consider the hysteric contribution associated with the long-wavelength surface roughness of the substrate. Next, we consider the role of adhesion between the rubber and the substrate and we show that this interaction will deform the rubber at the surface so that it will “follow” the short-wavelength surface roughness profile of the substrate. This gives an additional contribution to the sliding friction which we estimate. Finally, we discuss some other aspects of rubber friction and of the friction of other polymers sliding on hard substrates.

2. Rubber friction: the role of long-wavelength surface roughness

In this section we use the theory of viscoelasticity to estimate the contribution from the long-wavelength surface roughness of the substrate to the friction force when rubber is slid on a rough hard substrate.

The energy dissipation in a viscoelastic media is, in general, given by

\[ \Delta E = \int d^3x \, d\sigma_{ij} \dot{\epsilon}_{ij}, \]

where \( \sigma_{ij} \) is the stress tensor and \( \epsilon_{ij} \) the strain tensor. The dot denotes the time-derivative. For uniaxial deformations of a cylindrical bar this formula reduces to

\[ \Delta E = \frac{V}{2\pi} \int d\omega \, \text{Im} \left[ \frac{1}{E(\omega)} \right] |\sigma(\omega)|^2, \]

or

\[ \Delta E = \frac{V}{2\pi} \int d\omega \, \text{Im} \left[ \frac{1}{E(\omega)} \right] |\sigma(\omega)|^2, \]  

where \( V \) is the volume of the solid. Fig. 1a shows the real \( E_1(\omega) \) and imaginary part \( E_2(\omega) \) of the complex elastic modulus \( E = E_1 + iE_2 \) of rubber (schematic). Fig. 1b shows the the function \(-\text{Im} E(\omega)/|E(\omega)|\). One should note that the latter function is maximal for \( \omega = 1/\tau \), where \( \tau \) can be interpreted as the typical time associated with the flipping of some segment of a rubber molecule from one configuration to another. Note that the “flipping” is a thermally activated process and \( \tau \) depends exponentially (or faster) on the temperature, \( \tau \approx \tau_0 \exp(3k_B T) \).

We can use the equations above to estimate the contribution from the internal friction to the sliding friction of rubber. The sliding of rubber on a substrate results in fluctuating stresses characterized by the frequency \( \omega_0 \sim v/l \), where \( v \) is the sliding velocity and \( l \) a length of order the diameter of the contact area between a substrate asperity and the rubber surface. The main part of the energy “dissipation” occur in a volume element \( V \sim F^2 \), see Fig. 2. If \( F_n \) is the normal force, then the fluctuating stress \( \sigma \sim \sigma_0 \cos \omega_0 t \).
The fluctuating stress $\sigma(t)$ acting on the surface area $A \sim 1 \times 1$ gives rise to energy “dissipation” via the internal friction of the solid. The main part of the “dissipation” occurs in the dotted volume element $V \sim 1 \times 1 \times 1$.

$$\sigma_0 = F_0 \sqrt{2}. \text{ Substituting this in Eq. (1) gives}$$

$$\Delta E \approx \frac{1}{2} \sigma_0^2 \tau \Im \left[ \frac{1}{E(0_0)} \right],$$

where $T$ is the total time and the oscillating stress has acted on the solid. The energy dissipation per unit time $\Delta E/T$ must equal the product $F \nu$ between the friction force and the sliding velocity so that

$$F \approx \frac{\sigma_0^2 \nu}{\nu} \Im \left[ \frac{1}{E(0_0)} \right].$$

Since $F_0 = \nu \sigma_0$ we get

$$\mu = \frac{F}{F_0} \approx \sigma_0 \Im \left[ \frac{1}{E(0_0)} \right].$$  \hspace{1cm} (3)

We can estimate $\sigma_0$ as follows. In experiments with rubber it is usually found that the friction coefficient $\mu$ is nearly independent of the load $L$. (This is not true when the load $L$ is very small or very large. In the former case, adhesion between the rubber and the substrate becomes important. In the latter case, the area of real contact approaches the apparent area of contact and the area of real contact cannot increases linearly with the load.) This is the case, for example, in the experiments by Grosch [2] and by Mori et al. [7], and it is also expected from the model calculation of Greenwood, which shows that when two elastic solids (no plastic deformation) with rough surfaces are pushed together, the area of real contact and the average pressure $\sigma_0$ in the contact areas is nearly independent of the load. Since no plastic deformations are assumed to occur, the elastic modulus $E$ is the only quantity in the problem with the same unit as pressure, and it follows immediately that $\sigma_0 = CE$, i.e. the dimensional analysis alone predicts that the average pressure in the areas of real contact is proportional to the elastic modulus $E$ of the rubber, $C$ (which may be larger or smaller than unity) depends only on the nature of the surface roughness. The model of Greenwood [8] gives an explicit expression for $C$, but the only relevant result here is that for the very rough surfaces typically involved in rubber friction, $C$ is of order unity. For sliding surfaces $E(0_0)$ is complex and we will assume that $\sigma_0 \approx C|E(0_0)|$. Note that $\sigma_0$ depends on the sliding velocity $v$.

Increasing $v$ leads to a stiffening of the elastic properties and (for a given load) to a reduced contact area, and to an increased surface stress $\sigma_0$ in the contact areas. With this $\sigma_0$, Eq. (3) takes the form

$$\mu \approx -C \Im \frac{E(0_0)}{|E(0_0)|},$$  \hspace{1cm} (4)

Since $-\Im E(0_0)/|E(0_0)| \sim 1$ when $v_0 \sim 1/\tau$ (see Fig. 1b) one expects the maximum of the coefficient of friction to be of order unity.

Eq. (4) illustrates some interesting consequences of the internal friction. In some cases one wants the sliding friction to be as large as possible at some frequency $v_0$. This can be arranged by choosing a polymer characterized by a relaxation time $\tau$ such that $\varepsilon(0_0) \sim 1/\tau$, for which $-\Im E(0_0)/|E(0_0)|$ is maximal. In other cases, e.g. during the rolling of a wheel on the road, one wants the friction to be as small as possible. This can be arranged by using a material where, for typical rolling velocities, $v_0$ differs appreciably from $1/\tau$.

To illustrate the discussion above, Fig. 3 shows the friction coefficient at room temperature ($T = 20^\circ C$) for acrylonitrile–butadiene rubber sliding on a silicon carbide paper (solid curve) and on a smooth glass surface (dashed curve) [2]. The figure shows $\mu$ up to $v \sim 10^4$ m/s, but the actual measurements were only performed at low sliding velocities (up to $\sim 1$ cm/s), in order to avoid heating effects.
$E(v) = \frac{E(v)}{E(v)}$ of the rubber is maximal for $v = \frac{1}{t} \approx 10^5 \text{s}^{-1}$. If $\lambda$ denotes the wavelength of the characteristic surface roughness ($\lambda \approx 100 \text{Å}$ and $\approx 0.01 \text{cm}$ for the glass and silicon carbide surfaces, respectively), then the surface asperities will exert fluctuating forces on the rubber characterized by the frequency $v_0 = \frac{1}{\lambda} \approx v/\lambda$. The sliding friction coefficient will be maximal when $v_0 \sim \frac{1}{t}$, i.e. when $v = \frac{\lambda}{t}$, which gives $v \approx 10$ and $0.001 \text{m/s}$ for the silicon carbide and glass surfaces, respectively, which is in good agreement with the experimental data (see Fig. 3). [Note that the sliding friction for rubber on smooth surfaces, such as the glass surface in Fig. 3 (dashed line), is usually associated with an adhesional stick-slip process [3,4]. However, this mechanism does not give the correct temperature dependence of the friction force (see Section 5).]

3. Adhesion of rubber and its role in friction

The area of real contact and the sliding friction between rubber and a rough, hard substrate is affected by adhesion. In this section we discuss how adhesion affects sliding friction for elastically soft solids, such as rubber or gelatine.

For elastically hard materials, adhesion usually does not manifest itself on a macroscopic scale. For hard solids, the area of real contact consists of nearly randomly distributed contact areas (juncts), where surface asperities from the two surfaces “touch”. The surface asperities are elastically deformed and since they have different sizes and shapes, the junctions will “pop”, one after another, as the block is removed.

The situation is drastically different for rubber and other elastically soft solids, e.g. gelatine. In these cases, even a weak adhesive junction, e.g. resulting from the van der Waals interaction between the surfaces, may be elongated before breaking by a distance which is larger than the “height” of the surface asperities. Thus, at one stage during the removal of the block, a large fraction of the junctions will be simultaneously elastically elongated and exert a force on the block in the direction towards the substrate, see Fig. 4.
Fig. 4. Elastically deformed junctions during pull-off of an elastically soft solid from a hard substrate.

Furthermore, for an elastically soft solid, the area of real contact will, in general, be much larger than for an elastically hard solid under similar conditions. In particular, the area of real contact is nonzero even when the external load vanishes, which implies that the friction coefficient $\mu \rightarrow \infty$ as $L \rightarrow 0$. The effects discussed above also explains how some adhesives, e.g. tease film, function.

Let us study the area of real contact when an elastic body with a flat surface is squeezed against a rough surface of a hard solid, see Fig. 5a. Assume that the surface roughness is characterized by the height $h$ and the wavelength $l$. If the elastic solid completely fills out a "cavity", the elastic energy stored in the local deformation field will be of the order $E_{el} \approx Ghl^2$ (see Appendix A), where $G$ is the shear modulus, while the adhesion energy $E_{ad} \approx \Delta \gamma l^2$ will be gained ($\Delta \gamma$ is the change in the interfacial free energy per unit area when the two solids comes in contact). Thus, if the adhesion energy is just large enough to deform the solid, so that it completely fills out the cavity, then $E_{el} = E_{ad}$ or $h/l = (\Delta \gamma / G)^{1/2}$. For rubber at room temperature, $\gamma \approx 1000 \text{ kg/m}^2$ and $G \approx n_k T \approx 10^6 \text{ N/m}^2$ ($n$ is the number of rubber molecules per unit volume) and with $\Delta \gamma = 3 \text{ meV/Å}^2$ and $l = 1000 \text{ Å}$ this gives $h/l \approx 1$. We conclude that when rubber is slid at low speed on the polished glass surface in Fig. 3 (for which $l \approx h \approx 100 \text{ Å}$), the rubber in the contact area will deform (because of the adhesional forces) in such a manner as to completely follow the short-wavelength surface roughness profile of the glass substrate, see Fig. 5b.

At high sliding velocity, the rubber becomes stiffer and the area of real contact decreases.

Let us estimate the contribution to the friction from the adhesive interaction. We consider the sliding of rubber on a macroscopically smooth surface, polished by small particles with the (average) diameter $D$. Such a surface will have a roughness with the characteristic wavelength $l \sim D$ and height-fluctuation $h \sim D$. We assume that $D$ is so small that the rubber will completely fill out the cavities as a result of the adhesion force. As shown above, for rubber at room temperature, this requires that $D$ is smaller than $\sim 1000 \text{ Å}$.
Now, the perpendicular force, $F_n$ which must act on the surface of the rubber within the area $I \times l$ in order for the rubber to fill out a “cavity”, is $F_n \approx \frac{2 \pi h}{l}$ (note that the shear modulus $G$ is complex at finite frequencies). During sliding (velocity $v$), the rubber will be deformed elastically by the adhesional forces at the characteristic frequency $\nu_0 \sim \frac{v}{l}$. Thus, using Eq. (2), the total energy “dissipation” in the rubber during the time period $T$ and for the surface area $A$ (with $\sim A/l^2$ “cavities”) is

$$\Delta E \approx \left(\frac{A}{l^2}\right)^2 \left(\frac{F_n}{l^2}\right)^2 \left(\frac{v}{l}\right) T \operatorname{Im} (1/E)$$

$$= 4A \left(\frac{G}{l^2}\right)^2 \varepsilon \tau T \operatorname{Im} (1/E).$$

This must equal $\varepsilon \tau F$ so that the frictional stress $\sigma = F/A$ equals

$$\sigma = 4G^2 \left(\frac{h}{l}\right)^2 \operatorname{Im} (1/E) \approx -(\frac{h}{l})^2 \operatorname{Im} G(\nu_0),$$

(5)

since $G \approx 0.3E$ for rubber. At room temperature and for frequencies in the range $0.1 \text{ s}^{-1} < \omega < 10^8 \text{ s}^{-1}$, $-\operatorname{Im} G \approx 10^{-1} - 10^{-2} \text{ N m}^2$, so that $h \sim l$, $\sigma \sim 10^{-4} - 10^{-3} \text{ N m}$. This is of the same order of magnitude as the observed frictional stress in most cases [9]. It is interesting to note that many rubbers exhibit a relatively high internal friction even at frequencies as low as $0.1 \text{ s}^{-1}$. This corresponds to very low sliding velocities, $v = \omega l < 10^{-3} \text{ m/s}$ if $l < 1000 \text{ Å}$, and implies that the so-called static friction coefficient in most practical cases has a contribution from the internal friction of the rubber.

It is clear from the discussion above that if the adhesive interaction between the substrate and rubber can be reduced, the friction force will decrease. For example, if the glass surface, in the experiments reported above (see Fig. 3), is dusted with magnesia powder, thus preventing direct contact between the rubber and the track, the friction is found to remain almost constant over the whole range of temperatures and velocities, instead of the peak-structure (dashed line in Fig. 3) observed for the clean glass surface and associated with the internal friction of the rubber. The magnesia particles are embedded in the rubber surface (see Fig. 5c), leading to reduced adhesion and viscoelastic deformations of the rubber. On the other hand, the high-velocity peak in the friction coefficient for rubber sliding on the silicon carbide surface (solid line in Fig. 3) is not reduced for the dusted surface. This result is expected, since the long-wavelength deformations of the rubber, from which this peak is derived, are the same for the dusted and clean surfaces. Similarly, if rubber is slid on a substrate (assumed hydrophilic) covered by water, the sliding friction is reduced [9]. The water is trapped in the surface cavities of the substrate, leading to reduced viscoelastic deformations of the rubber (see Fig. 5d). Further proof of the important role of adhesion on sliding friction is found to remain almost constant over the whole range of temperatures and velocities, leading to reduced adhesion and viscoelastic deformations of the rubber. The argument due to Greenwood then shows that
4. Discussion

We have seen that when rubber is slid on a hard substrate, with a surface roughness characterized by a (single) length scale $l$, very high friction occurs when $\tau \approx \tau^*$, where $1/\tau^* = v/l$ is the frequency of the fluctuating forces on the rubber, and $1/\tau$ is the frequency for which the internal friction of the rubber is maximal. Most other polymers exhibit much lower sliding friction than rubber; one extreme case being Teflon for which the static and kinetic friction coefficients are smaller than 0.1. This implies that for most polymers the internal friction is much smaller than with the aluminum substrate. In particular, for the low energy rubber surface 3 the coefficient of friction for the Teflon substrate did not change with load above 0.2 N.

Figure 6. Effect of the surface free energy of rubber on the relation between the friction coefficient and load for (a) an aluminum substrate and (b) a Teflon substrate (from [7]).

Figure 7. Rubber sliding on a hard substrate having short-ranged and long-ranged surface roughness. Because of the adhesional interaction the rubber deforms and fills out the "cavities" associated with the short-ranged surface roughness.

The area of real contact is proportional to the load. However, adhesion is very important in each asperity contact area associated with the long-wave-length roughness, as it will force the rubber to fill out all the small-sized (say $D < 1000 \text{ Å}$) "cavities" of the substrate in each such contact region, see Fig. 7. The load at which the coefficient of friction becomes constant, increases with increasing surface free energy. It should also be noted that, as expected, the coefficient of friction increases with increasing surface free energy of the rubber. This shows that the adhesional interaction contribute in an important manner by locally deforming the rubber in each junction as indicated in Fig. 7. For the Teflon substrate, the adhesion to rubber is smaller than with the aluminum substrate. In particular, for the low energy rubber surface 3 the coefficient of friction for the Teflon substrate did not change with load above 0.2 N.

4. Discussion

We have seen that when rubber is slid on a hard substrate, with a surface roughness characterized by a (single) length scale $l$, very high friction occurs when $\tau \approx \tau^*$, where $1/\tau^* = v/l$ is the frequency of the fluctuating forces and the substrate asperities exert on the rubber, and $1/\tau$ is the frequency for which the internal friction of the rubber is maximal. Most other polymers exhibit much lower sliding friction than rubber; one extreme case being Teflon for which the static and kinetic friction coefficients are smaller than 0.1. This implies that for most polymers the internal friction gives a much smaller contribution to the sliding friction than for rubber. The physical origin of this can be understood as follows.

Note first that $1/\tau$ is the rate by which some (specific) molecular segments in the polymer switches between different configurations, say between two states $A$ and $B$, separated by an energy barrier of height $\epsilon$. Thus, $\tau \sim \exp(\epsilon/k_B T)$ depends exponentially on the inverse of the temperature. Now, for most polymers the activation energy $\epsilon$ is so large that, at room temperature, $\tau$ is so large that for typical sliding velocities $\dot{\tau} \ll \tau$ and the internal friction can be neglected. Note,
however, that at high enough temperature one may have $T \approx T_g$, in which case the internal friction is important.

A measure of the activation energy $E$ is the glass transition temperature $T_g$. For most types of rubber, $T_g \approx 200$ K. On the other hand, for many other (glassy) polymers $T_g \approx 350$ K, and in these cases the internal friction gives a negligible contribution to the sliding friction under typical sliding conditions.

When the contribution from the internal friction is small, the friction force will be determined mainly by an interfacial process, in most cases probably involving the shearing of a molecular thin organic contamination layer (this process may also be the origin of the flat “background” contribution to the friction coefficient seen in Fig. 3 away from the resonance peak; see Section 5). Indeed, the available experimental evidence on the frictional behavior of most polymers, other than rubber, indicate that the superposition principle discussed above (which implicitly assume that the sliding friction is dominated by the internal friction) is not applicable to these polymers.

Since most polymers are soft, it is unlikely that the organic contamination layer will be squeezed out by the relative low pressure which occurs in the contact areas. The friction coefficient, when a steel ball is slid on most polymers at “low” sliding velocities (to avoid heating effects), is in the range $0.3 \pm 0.1$ and probably reflects shearing of the contamination layer. At “high” sliding speeds, because of the low thermal conductivity and low melting point of most polymers, melting of the polymer usually occur, resulting in a rapid drop in the friction force with increasing sliding velocity (just as in frictional melting of ice), and to large wear. The spectacular low friction of Teflon is probably related to its low surface energy (the Teflon surface is not wetted by most organic liquids).

Finally, we note that because of its low elastic modulus, rubber often exhibits elastic instabilities during sliding. The most well-known involves the compressed rubber surface in front of the contact area undergoing a buckling, producing detachment waves which propagate from the front- to the back-end of the contact area. These so-called Schallamach waves were first discovered in 1971 and have been intensively studied [10].

5. Stick–slip molecular mechanism of rubber friction

It is clear that, on a perfectly smooth substrate, the two frictional processes discussed above are absent. However, in the literature another contribution to rubber friction from adhesion has been proposed [3,4], which is assumed to contribute in this latter case. It is assumed that, during sliding, a thermally-activated molecular stick–slip process occurs in the rubber at the rubber–substrate interface. For example, a segment of a rubber molecule may be temporarily pinned on the substrate, followed by the stretching or elongation of the rubber molecule, and a rapid “snapping” or “flipping” of the molecular chain, as the bond between the rubber molecule and the substrate is broken (see Fig. 8). The breaking of the bond is the combined result of the stretching force and thermal excitation. Thus, the temperature dependence of this process is not determined by that of the internal friction of the rubber, but by the activation energy necessary to break the bond between the rubber molecule and the substrate. However, the experiments by Grosch, for rubber sliding on glass, show that the friction force has the same temperature dependence as that of the internal friction of the rubber.

Another objection to this theory is that most real surfaces are covered by thin organic contamination layers. Such layers automatically form in the normal atmosphere (experiments have shown

![Fig. 8. The interaction between rubber molecules and a hard substrate during sliding in the model study of [3] (schematic).](image-url)
that metal surfaces cleaned in ultra-high vacuum will be covered with a $\sim 10$ Å thick organic contamination layer if kept in the normal atmosphere for a few hours), but is also expected as a result of the wear of the rubber surface during sliding. It is clear that a direct interaction between the rubber molecules and the substrate will not occur in most cases, but the rubber will interact with the (mainly physisorbed) contamination layer. During sliding it is likely that the rubber at the interface undergoes local stick-slip motion, while the contamination layer fluctuates between a solid (probably “glassy”) state at stick (which locally pins the rubber surface to the substrate) and a fluidized state during slip. This type of local stick-slip motion is expected to occur also for other lubricated sliding systems at low sliding velocities (see [11,12]). The contribution to the friction force from this mechanism is expected to be only weakly velocity (and temperature) dependent, and will not exhibit the same temperature dependence as the internal friction of rubber. It may, however, be the origin of the background contribution to the friction force seen away from the peak-structure in Fig. 3 (glass substrate). As pointed out before, for very smooth rubber surfaces, so-called Schallamach waves have been observed under quite general conditions [10]. These waves can be considered as resulting from elastic instabilities, and give a friction force which is nearly velocity independent.

6. Summary and conclusion

We have presented a new theory of rubber friction. In accordance with earlier work we attribute a large fraction of rubber friction to the energy “dissipation” occurring in the bulk of the rubber (caused by the internal friction of the rubber), due to the fluctuating surfaces stresses acting on the rubber from the surface asperities of the (hard) substrate. Our analysis of this problem resulted in a general formula for rubber friction (Eq. (4)). We have also shown that the adhesive interaction between the rubber and the substrate (for clean surfaces), will induce additional viscoelastic deformations of the rubber in such a manner that (at low sliding velocity) the rubber will completely “follow” or “fill-out” the short-wavelength surface roughness profile of the substrate. This gives an additional contribution to the friction force which has been estimated (Eq. (5)).

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Appendix A

We assume that isotropic elastic media occupy the half space $z>0$. On the surface $z=0$, it is assumed that a perpendicular stress, $\sigma$, acts within a small circular region with radius $R$, that is centered at a material point $P$. This gives rise to an elastic deformation of the solid and $P$ will be displaced perpendicular to the surface by the amount (see Ref. [13])

$$\frac{1-n^2}{E} \sigma \int_{r=R} d^2x \frac{1}{r},$$

where $r^2=x^2+y^2$ and where $\nu$ is Poisson’s ratio and $E$ Young’s modulus. The integral is trivial to perform, giving

$$u=2(1-n^2) \sigma R/E.$$

If $D=2R$ is the diameter of the contact area and $F=\pi R^2 \sigma$ the total force, we get

$$F=(\pi D/4) E u/(1-n^2)=(\pi D/2) G u/(1-\nu). \quad (A.1)$$

References