WAVE PROPAGATION IN CONTINUOUS PERIODIC STRUCTURES: RESEARCH CONTRIBUTIONS FROM SOUTHAMPTON, 1964–1995

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(Received 1 November 1995)

After brief reference to some early studies by other investigators, this paper focuses mainly on methods developed at the University of Southampton since 1964 to analyze and predict the free and forced wave motion in continuous periodic engineering structures. Beginning with receptance methods which have been applied to periodic beams and rib–skin structures, it continues with a method of direct solution of the wave equation. This uses Floquet’s principle and has been applied to beams and quasi-one-dimensional periodic plates and cylindrical shells. Sample curves of the propagation and attenuation constants pertaining to these structures are presented. A limited discussion of the transfer matrix then follows, after which the method of space-harmonics is introduced as the method best suited to the prediction of sound radiated from a vibrating periodic structure. Reviewed next are some theorems and variational principles relating to periodic structures which have been developed at Southampton, and which form a basis for finding natural frequencies of finite structures or for computing free and forced wave motion by energy methods. This has led to the finite element method (in its standard and hierarchical forms) being used to study wave motion in genuine two-dimensional and three-dimensional structures. Examples of this work are shown. The method of phased array receptance functions is then introduced as possibly the easiest way of setting up exact equations for the propagation constants of uniform quasi-one-dimensional periodic structures. A summary is finally presented of the limited and early work performed at Southampton on simple disordered periodic structures.

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1. INTRODUCTION

Ellyn Richards’ early vision and pioneering zeal in the study of aeroplane noise at Southampton had a “Coanda effect” upon those of us involved in structural teaching and research. We were inexorably drawn into the study of structural vibration caused by the noise of the early jet engines, which were shaking and shattering flimsy aeroplane structures to pieces. Something had to be done about it! While quieter engines were yet to be developed, less responsive and more fatigue-resistant structures had to be designed and “EJR” gave much encouragement to three of us to work to this end—B. L. Clarkson, the late T. R. G. Williams and myself. His reputation and fund-raising ability drew the attention of the U.S. Air Force, which awarded us generous grants for vibration and

fatigue research. His energy and detailed planning led us, in 1959, through an exhausting but eye-opening tour of the associated research activities in U.S.A., a tour which concluded with the 1st International Conference on Acoustic Fatigue. In the relaxed atmosphere of this conference, his gifts as a humorist and singer were acclaimed, in addition to his gift of research leadership!

Much more could be said about his warmth and friendliness, his concern for the personal advancement of the members of his team, his sense of humour which never resented his leg being pulled and often led him to stretch mine! But this is meant to be a scientific paper, written in his 81st year to acknowledge his great contribution to noise and vibration research. It outlines one of the strands of vibration research which has continued to unite the Department of Aeronautics and Astronautics and the Institute of Sound and Vibration Research at Southampton long after they became separate units in 1963.

A periodic structure consists fundamentally of a number of identical structural components ("periodic elements") which are joined together end-to-end and/or side-by-side to form the whole structure. The atomic lattices of pure crystals constitute perfect periodic structures, but these are lumped parameter systems with discrete masses (the atoms) interconnected by the inter-atomic elastic forces. In structural engineering the mass and elasticity of structural members are continuous, and constitute periodic structures when arranged in regular arrays.

Engineering structures which are, or have been treated as, periodic include multi-storey buildings, elevated guideways for high speed transportation vehicles ("Maglev" systems), multi-span bridges, multi-blade turbines and rotary compressors, chemical pipelines, stiffened plates and shells in aerospace and ship structures, the proposed space station structures and layered composite structures. In the design of these structures, account must be taken of the vibration levels likely to be caused by the time dependent forces, pressures or motions to be encountered in service life. Buildings may experience earthquake excitation and the periodic forces of reciprocating or rotating machinery. Elevated guideways and bridges are subjected to the moving weight of vehicles, turbines and compressors to turbulent aerodynamic flow and sundry instabilities, and chemical pipelines reciprocating pump forces or to hydrodynamic forces from internal moving fluids. Aeroplane structures are subjected to random convected pressure fields from jet noise at low speed and turbulent boundary layers at high speed. Ship structures are excited by engine vibration. The proposed space station will be subjected to impulsive forces from control thrusters and docking impacts and also to periodic forces from rotating machinery. Whatever the nature of the forcing function, wave motion is generated within the structure. The associated levels of vibration and shock response must be predictable in order that the structure can be designed with a minimum probability of catastrophic damage or malfunction in service.

Periodic structures may be categorized in different ways. They may be one-, two- or three-dimensional. They may consist of beam, bars, flat plates or curved shells in various combinations and with different support conditions. Their applied time-dependent loadings may be localized or widely distributed, harmonic or random, short-term impulsive or longer-term transient. The significant response quantity for one structure may be the maximum bending moment; for another it may be the radiated or transmitted sound power and for another the vibrational power flow from a source or the surface acceleration at a remote point.

While the simplest structures transmit vibrational energy by just one type of wave motion (flexural waves, say), others transmit it in simultaneous and particular combinations of longitudinal, torsional and bi-directional flexure. When these different wave types encounter a discontinuity in the periodicity, they interact and are converted
from one type into another. This is an important part of the wave propagation process in the very large space station structures.

Many significant papers have been published on periodic structures, but in view of the occasion of this paper its principal focus will be on the contributions from the research groups at the University of Southampton. Where desirable, limited reference is also made to the work of other researchers.

2. PERIODIC STRUCTURE ANALYSIS UP TO 1965

Brillouin [1] traced the history of the subject back over 300 years to Sir Isaac Newton, but until 1887 the systems considered were lumped masses joined by massless springs. These were sufficient to enable the ideas of free wave propagation in such structures to be developed. These waves have propagation constants in the dual form of an attenuation constant \( \mu \) and a phase constant \( \epsilon \). At any frequency, the motion of one element of a harmonically vibrating infinite system is equal to \( e^{\mu x} \) or \( e^{\epsilon x} \) times that of its neighbour, the plus or minus sign depending on the direction of the wave motion.

Rayleigh made the first study of a continuous periodic structure in 1887 [2], considering a stretched string with a periodic and continuous variation of density along its length and undergoing transverse harmonic vibration. The governing wave equation is of second order with a periodic coefficient and Rayleigh solved it by Hill’s method. He found the phase velocities of propagating waves and the spatial decay factors in attenuating waves. His work is applicable to any simple periodic structure the wave motion of which is governed by a second order differential equation.

Between 1900 and 1960, mathematical techniques were developed for analyzing increasingly complicated crystal lattice structures, periodic electrical circuits and continuous transmission lines. Many of these techniques have been invaluable in subsequent studies of continuous periodic engineering structures. Cremer and Leilich [3] used some of them in 1953 to study harmonic flexural wave motion along a one-dimensional periodic beam either with simple supports or with point masses at regular intervals \( L_x \). With simple supports it constitutes a “mono-coupled” periodic system, as its basic periodic element (a single beam-bay on simple supports) is coupled to each of its neighbours through just one displacement co-ordinate. At any frequency it therefore has just one pair of equal and opposite propagation constants, \( \pm \mu_x \) or \( \pm \epsilon_x \), and these were found to be given by the following linear equation in \( \cosh \mu_x \):

\[
\cosh \pm \mu_x = \frac{\cos \beta L_x \sinh \beta L_x - \sin \beta L_x \cosh \beta L_x}{\sinh \beta L_x - \sin \beta L_x} = \cos \pm \epsilon_x.
\]  

(1)

\( \beta \) is the flexural wavenumber of the uninterrupted uniform beam at the frequency \( \omega \). \( \beta^2 = \omega^2 \rho A/EI \), where \( \rho A \) is the mass per unit length of the beam and \( EI \) is its flexural rigidity.

Cremer and Leilich presented curves of \( \mu_x \) and \( \epsilon_x \) versus \( \omega \) for the undamped simply supported beam in the form shown on Figure 1(a). The practice at Southampton has been to present them in the more compact form of Figure 1(b), it being recognized that both positive and negative values of \( \epsilon_x \) are admissible and that \( \pm 2n\pi \) may be added to either of these. The continuous periodic structure has an infinite number of alternating attenuation zones and propagation zones. This distinguishes it from the lumped mass systems previously investigated, for a system with \( N_{\text{ dof}} \) degrees of freedom in each periodic element (e.g., \( N_{\text{ dof}} \) different lumped masses) has only \( N_{\text{ dof}} \) propagation zones. In the attenuation zones the flexural motion of the beam decays along the beam length and one such wave on its own can transmit no energy. In the propagation zones the flexural motion
is that of a genuine propagating wave (albeit of complicated form) which does transmit energy. The uniform beam with periodic point masses has two coupling co-ordinates between each pair of periodic elements—the flexural rotation and the flexural displacement. This is a bi-coupled periodic system, and at any frequency has just two pairs of equal and opposite propagation constants. These can be found from a quadratic equation in \( \cosh \mu \) [3].

In 1956, Miles [4] sought the natural frequencies of a finite periodic uniform beam resting on an arbitrary number of simple supports (see Figure 2(a)). This mono-coupled system has a single internal moment and rotational co-ordinate between one element and the next.
(see Figure 2(b)), and at three successive junctions the harmonically varying moments are \(M_{n-1}, M_n\) and \(M_{n+1}\), and the rotations (flexural gradients) are \(\theta_{n-1}, \theta_n\) and \(\theta_{n+1}\). For continuity of rotation across each support, the moments must satisfy the dynamic three-moment equation

\[
M_{n-1} - M_n \left(\frac{(a_{RR} + a_{LL})}{a_{RL}}\right) + M_{n+1} = 0, \tag{2}
\]

where the \(a\)'s are the transcendental frequency dependent receptance functions.

An identical equation applies to each support junction, so the whole set of recurrence equations is satisfied by \(M_n = M' \sin (n \omega x), M_{n+1} = M' \sin [(n+1) \omega x]\) and \(M_{n-1} = M' \sin [(n-1) \omega x]\). Equation (2) then yields

\[
\cos \omega x = \left(\frac{a_{RR} + a_{LL}}{2a_{RL}}\right), \tag{3}
\]

which is actually a more general form of equation (1). Miles showed that the natural frequencies of a finite beam with \(n\) bays and \(n+1\) simple supports are those frequencies at which \(\omega, n=0, \pi, 2\pi, 3\pi, \ldots\) etc. These frequencies are found by numerical solution of equation (3) when the corresponding values of \(\omega x\) are inserted. Miles proceeded briefly to investigate the phase and group velocities of the constituent waves.

In 1964, Heckl [5] investigated a two-dimensional periodic structure consisting of a rectangular grillage of interconnected uniform beams which had both flexural and torsional stiffness. In his high frequency analysis he considered the multiple reflection and transmission processes as flexural waves in one beam element impinge on the junctions with adjacent beams. An equation for the propagation constants was established in terms of the reflection and transmission coefficients which relate to a single wave in just one infinite beam when it impinges on the junction with just one other infinite beam. The analysis is relatively unwieldy compared with the exact methods now available to deal with such a structure, but the equations that it yields for the propagation constants are relatively simple, albeit approximate. It may be noted that the use of reflection and transmission coefficients was subsequently revived by Hodges [6] when he considered disordered periodic systems in 1982.

Ungar [7] examined the steady state harmonic responses and propagation constants of a one-dimensional periodic beam made periodic by the attachment of arbitrary but identical non-dissipative impedances at regular intervals. Adopting Heckl's assumptions and method of multiple reflections, he found the propagation constants and response of the beam when excited harmonically between the impedances. He also made an exact analysis (not restricted to very small wavelengths) for a beam harmonically loaded at one or all of the impedance locations.

3. RECEPTANCE METHODS APPLIED TO PERIODIC STRUCTURES, 1964–1970

Periodic structures were first studied at the University of Southampton in 1964 in the context of noise-excited vibration and sonic fatigue of stiffened aerospace structures. Large areas of these consist of uniform plates and shells with identical stiffeners at regular intervals, and research into their natural frequencies, modes and random response levels was required with a view to predicting stress levels and fatigue endurance. In particular, the influence of heavy damping was to be investigated as a means of reducing stress levels. It was felt that if the damping levels were high enough, the general response levels would be the same whether the structure was finite or infinite, exactly periodic or slightly disordered. This feeling was subsequently justified by our own calculations and also by those of Lin [8], Lust et al. [9] and others. As we proceeded, we found that periodic
structure theory was well suited to lightly damped as well as to heavily damped finite periodic structures.

In this early work, a two-dimensional stiffened plate had to be reduced for analytical purposes to an equivalent “quasi-one-dimensional” stiffened plate. It was supposed that a section of plate between an adjacent pair of the stiffest stiffeners (the \( x \)-wise set, say) could be treated in isolation and that these stiffeners provided ideal simple supports to the plate. The spatial variation of the plate motion between these stiffeners (in the \( y \)-direction) was therefore assumed to be sinusoidal with an integral number of half sine waves, i.e.,

\[ w(x, y) = w(x) \sin n \pi y / L_y. \]

† The same assumption was being made at that time for these structures by Lin et al. [11] (who went on to use transfer matrix methods of analysis) and by Clarkson et al. [12] who were computing natural modes and frequencies.

The plate was periodic in the \( x \)-direction by virtue of the stiffeners at the intervals \( L_x \) and its wave motion was governed by a reduced form of the standard fourth order flexural wave equation. This reduction allows a whole range of analytical methods to be used to study the motion, including those of direct solution, receptances, transfer matrices and approximate energy methods.

In our first paper in 1965 (by Mead and Wilby [10]), receptance functions were used to set up recurrence equations of the form of equations (2). These expressed continuity of the plate flexural gradient and flexural displacement at the periodic line-junctions created by the \( y \)-wise set of periodic stiffeners. The moments and shear forces along the \( y \)-wise edges of one periodic element were of the form

\[ M_n(0, y) = M_n(0) \sin n \pi y / L_y, \quad S_n(0, y) = S_n(0) \sin n \pi y / L_y. \]

The recurrence equations related the edge moments and shear forces at three successive periodic line junctions on the plate, and these equations involved receptances such as

\[ a_{LL} = w_n(0) / S_n(0) \quad \text{and} \quad a_{L' L'} = w_n'(0) / M_n(0). \]

The effects of the stiffener rotational stiffnesses and transverse flexibilities were readily included in the receptances. The effect of structural damping was included by allowing flexural stiffnesses to take the complex forms

\[ D(1 + i \eta \rho) \quad \text{or} \quad EI(1 + i \eta \beta). \]

Attenuation and phase constants for the wave motion were obtained from the recurrence equations in the same way as that of Miles and were computed for different values of the stiffener rotational stiffness and the damping loss factor, \( \eta \). The effect on \( \mu \) and \( \xi \) of increasing the rotational stiffness when \( \eta = 0.025 \) is demonstrated in Figure 3. The lower bounding frequency of a propagation zone is seen to increase as the rotational stiffness increases. The upper bounding frequency was found to drop as the transverse stiffness decreases, but is not shown here.

Also investigated was the response of this structure to a single-point harmonic force in just one loaded bay. The motion of the loaded bay, de-coupled from the rest of the structure, was first evaluated. The flexural gradients and displacements at its ends were then made compatible with the free wave motion generated in the two unloaded semi-infinite periodic structures on either side of it. In Figure 4 it is shown how the driving point response in an infinite structure varies with frequency and stiffener rotational stiffness when \( \eta = 0.025 \). The peak response levels for the infinite periodic structure were found to be inversely proportional to \( \eta^{1/2} \). With \( \eta > 0.1 \) they were almost the same as those for a five-bay finite structure. The response to harmonic forcing was then used to find the response to random forcing of known spectral density. This showed that increasing the rotational stiffnesses of the stiffeners (which reduced the width of the propagation zones) increased the random response level in the loaded bay but decreased it in the adjacent unloaded bays. This corresponds to the general rule that the narrower the propagation band, the higher is the response due to distributed random excitation.

E. G. (Emma) Wilby conducted many other investigations into these periodic plates. They included calculations of propagation and attenuation constants for different combinations of the torsional and flexural stiffnesses of the \( y \)-wise stiffeners. Calculations
were also made of the random forced response of both finite and infinite plates excited by frozen convected pressure fields and convected boundary layer pressure fields. Conditions for large “coincidence type” responses were identified at which the convection velocity of the harmonic pressure field was equal to the phase velocity of free waves of the same frequency in the plate. This was all reported in her draft Ph.D. thesis [13] which, regrettably, was never formally submitted due to her emigrating to the U.S.A. with her
husband John. In its handwritten form it has been much appreciated by subsequent generations of research students. Her important contributions are acknowledged here.

In 1970, Sen Gupta presented his Ph.D. thesis [14] on wave propagation in beams, rib–skin structures and orthogonally stiffened flat plates. Rib–skin structures consist of a pair of parallel plates of possibly unequal thickness joined together periodically by another set of orthogonal finite plates (see Figure 5(a)). Sen Gupta allowed flexural rotation but no transverse displacement at each pair of line-junctions between adjacent periodic elements. Sinusoidal displacement modes in the $y$-direction were assumed in order to make the structure quasi-one-dimensional and the four moments and rotations at the two ends of a single periodic element (see Figure 5(b)) were related through the rotational receptances of the element at its junction lines. This yields

$$
\begin{bmatrix}
\beta_{ll} & \beta_{lr} \\
\beta_{rl} & \beta_{rr}
\end{bmatrix}
\begin{bmatrix}
M_l \\
M_R
\end{bmatrix}
=
\begin{bmatrix}
\theta_L \\
\theta_R
\end{bmatrix},
$$

(4a)

where

$$
\{M_l\} = \begin{bmatrix} M_{LT} \\ M_{LB} \end{bmatrix},
\{M_R\} = \begin{bmatrix} M_{RT} \\ M_{RB} \end{bmatrix},
\{\theta_L\} = \begin{bmatrix} \theta_{LT} \\ \theta_{LB} \end{bmatrix}
\text{ and }
\{\theta_R\} = \begin{bmatrix} \theta_{RT} \\ \theta_{RB} \end{bmatrix},
$$

(4b)

Each of the $\beta$’s in equation (4a) is a $2 \times 2$ rotational receptance matrix. The whole $4 \times 4$ matrix in the equation is symmetric so $\{\beta_{rl}\} = \{\beta_{lr}\}^T$. Floquet’s principle (sometimes called Bloch’s theorem [1]) was then invoked to make it apply to free wave motion in the whole infinite periodic structure with a propagation constant $\mu_\omega$ (or $i\omega_\omega$). Continuity
of $x$-wise flexural gradient and equilibrium of moments was then automatically ensured across all periodic junctions without recurrence equations actually being set up. Floquet’s principle for the rib–skin structure requires that $\{\theta_R\} = \exp m \{\theta_L\}$ and $\{M_L\} = -\exp m \{M_R\}$. Equations (4a) then reduce in order, to

$$[\beta_{RL} - e^{m\beta_{LL}} + e^{2m\beta_{LR}}] \{M_L\} = 0$$

or

$$[e^{-m\beta_{RL}} - [\beta_{LL} + \beta_{RR} + e^{m\beta_{LR}}] \{M_L\} = 0.$$ (5)

Equating the determinant of the matrix in this to zero leads to a quadratic equation for $\cosh m$. Sen Gupta therefore found two pairs of propagation constants at each frequency. From some of the computed values it was evident that two pairs of propagating waves (as distinct from attenuating waves) can exist in some frequency ranges [15], while in other ranges there may be only one or none at all.

He proceeded to show how the natural frequencies of finite one-dimensional or quasi-one-dimensional periodic structures (rib–skin, beam or stiffened plate) can be found from a graphical construction on the curves of $\nu$ versus frequency [16]. This method has since been incorporated in commercially available (ESDU) data sheets [17]. If such a structure has $n_i$ periodic elements, the frequencies at which $\nu = j\pi/n_i$, ($j = 0$ or 1 to $n_i - 1$ or $n_i$) are the natural frequencies of the finite system provided its extreme ends are fully fixed or simply supported. Sen Gupta proved this by application of the phase closure principle, but a more general proof was later presented by Mead for mono-coupled systems [18] and then for multi-coupled systems [19].

The response of the rib–skin structure to a point harmonic load or convected harmonic pressure field was also investigated by Sen Gupta, as well as the influence on the response of damping in the various components [14]. The earliest investigations into wave motion in a periodic two-dimensional plate were made also at this time, but only for a uniform plate on a rectangular array of simple supports. Rayleigh’s quotient and energy methods were not yet employed, and Sen Gupta’s method for this analysis was relatively tedious. Nevertheless, the results showed how the propagation constants of “plane” harmonic waves† in the two-dimensional plate vary with direction of propagation $\theta$ across the plate. The pass-bandwidth of the first propagation zone was shown to be greatest when the direction is perpendicular to the diagonal of the rectangle of the periodic element. This has important implications for the response of the plate when it is excited by a pressure field with components convecting in many different or random directions.

4. DIRECT SOLUTIONS FOR THE RESPONSE AND PROPAGATION CONSTANTS OF ONE-DIMENSIONAL PERIODIC STRUCTURES

Another method of analyzing wave motion in periodic structures was presented in 1971 [20], and eliminates the need to find the receptance functions or transfer matrices for the periodic elements. Although it was first formulated for beams, it is also applicable to any one-dimensional or quasi-one-dimensional continuous periodic structure. The response of a uniform beam to an infinitely extended convected harmonic pressure field $p(x, t) = p_0 \exp(\nu \omega t - k_x x)$ was first determined, the motion of the beam being

† A plane harmonic wave in a two-dimensional periodic system has the phase constant $\nu_x$ in the $x$-direction and $\nu_y$ in the $y$-direction. The direction of propagation of the wave relative to the $x$-axis is then given by $\theta = \tan^{-1} \left\{ [\nu_y/L_y] / [\nu_x/L_x] \right\}$. 

governed by the classical forced flexural wave equation $EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = p_0 \exp(i\omega t - \bar{k}_x x)$, which has the solution

$$w(x, t) = \sum_{n=1}^{4} A_n e^{\beta_n x} + \frac{p_0 \exp(-i\bar{k}_x x)}{EIk_\rho - \rho A\omega^2} \exp(i\omega t).$$  \hspace{1cm} (6)$$

The $\beta_n$'s are the four complex fourth roots of $\rho A\omega^2/EI$. The responses in all adjacent pairs of beam bays must be identical apart from the imposed phase difference of $\bar{k}_x L_x$ from bay to bay. The displacements, flexural rotations, moments and shear forces at the two ends of each periodic beam element bay are also the same apart from this phase difference, and these four relationships constitute four “wave-boundary conditions” which are required to find the four $A_n$'s. They lead to a matrix equation for the $A_n$’s which has the form

$$[E, \beta_n, k_\rho] \{A_n\} = \{P\} p_0/(EIk_\rho - \rho A\omega^2).$$ \hspace{1cm} (7)$$

The $4 \times 4$ matrix $[E, \beta_n, k_\rho]$ and the four-element column matrix $\{P\}$ are defined in reference [20]. Their elements are linear functions of the stiffnesses of the beam and its periodic elastic supports and of $\exp(\beta_n L_x)$, $\exp(k_\rho L_x)$, $\beta_n$ and $k_\rho$.

Equation (7) can obviously be used to find the $A_n$’s (and hence the beam response) without first finding any receptance functions, transfer matrices, etc. In Figure 6 is illustrated a set of frequency–response curves thus obtained for a damped simply supported beam subjected to pressure fields of different non-dimensional convection velocities, $CV = (\omega/k_\rho) \sqrt{\rho A\omega^2/EI}$. The frequency range shown is that of the first propagation zone, and each curve has no more than one major peak within it. The frequency at which the peaks occur can be identified with the frequency of one of the free waves in the periodic beam which has a phase velocity equal to the convection velocity of the pressure field.

This response of the infinite beam may be used as one component of the response of a finite beam to the same convected loading. The other components are the free waves of the periodic structure, which are reflected from the extreme ends of the beam and these can be found by appropriate satisfaction of the boundary conditions at these ends [20]. Computed results for a five-bay beam with a loss factor of 0·25 showed that its maximum response did not substantially exceed that of the infinite beam. The infinite periodic beam had the lower response, and one may conclude from this and from much other work that

![Figure 6. The frequency response at a bay centre of a periodic beam excited by a convected harmonic pressure field; the effect of convection velocities; beam loss factor = 0·025; \(CV = 2\); \(CV = 4\); \(CV = 8\); \(CV = 16\); \(CV = 32\); \(CV = \text{non-dimensional convection velocity}\).](image)
modifications to the infinite and periodic nature of the beam (by making it finite or by disordered its periodicity) lead to higher responses in at least part of the modified system. This can be explained by the existence of either local resonances in the disorders or overall resonances in the whole finite structure.

Experimental verification of the validity of periodic structure theory was obtained in 1972 by O'Keefe in his M.Sc. dissertation [21]. He used the direct solution for the forced wave motion and computed the response of a six-bay periodically stiffened plate to a grazing incidence random sound field. The heavily damped plate had modal loss factors of the order $\eta = 0.25$. Computed r.m.s. stresses in the plate exceeded measured values by about 30%. This difference must be regarded as very small when the simplifying assumptions of the plate theory are borne in mind.

The above analysis involving the $A_n$'s can be modified to yield equations for finding the propagation constants and characteristic free wave motions of the periodic beam [20], again without first finding receptances, transfer matrices, etc. The pressure amplitude is set to zero, and Floquet's theorem is used to set up the wave-boundary conditions; i.e., the state vector at the left hand end of one beam bay (in terms of the $A_n$'s) must be equal to $e^m$ times the state vector at the left-hand end of the next bay. Upon satisfying the necessary continuity and equilibrium conditions at the junction of the two bays, one then obtains the $4 \times 4$ matrix equation

$$[E_L, \beta_n] \{A_n\} = e^m [E_R, \beta_n] \{A_n\} \quad \text{or} \quad [E_R, \beta_n]^{-1} [E_L, \beta_n] \{A_n\} = e^m \{A_n\}. \quad (8a, b)$$

The matrices $[E_L, \beta_n]$ and $[E_R, \beta_n]$ are defined in reference [20]. $\mu$ is now found from the eigenvalues, $e^m$, of the matrix product in equation (8b) and the corresponding eigenvector $\{A_n\}$ can be used to determine the waveform $w(x)$ of the wave.

This basic method can be used to compute the free wave motion in any one-dimensional or quasi-one-dimensional continuous periodic system, and has been applied much more recently by Mead and Bardell to uniform cylindrical shells with periodic stiffening either around the circumference or along the length [22, 23]. The shell motion is governed by a differential equation of eighth order and is characterized by eight $A_n$'s, and this leads to $[E, \beta_n]$ matrices of order $8 \times 8$. The quasi-one-dimensional periodic shell has four coupling co-ordinates between adjacent periodic elements, and therefore has four pairs of propagation constants and corresponding free waves at each frequency. The four propagation constants obtained in this way, for a particular circumferentially stiffened shell, are shown in Figure 7.

5. TRANSFER MATRICES AND PERIODIC STRUCTURES

Links between the vibration groups at Southampton and Y. K. Lin’s group at the University of Illinois had been established in the early 1960s and both groups profited from the interaction. Lin and his co-workers were pioneering the application of transfer matrices to the analysis of stiffened plate vibrations and periodic structures [24–26], emphasizing that the method applies strictly to one-dimensional or quasi-one-dimensional systems. Mercer and Seavey [27] at Southampton had also used transfer matrices to compute natural frequencies and modes of stiffened plates, but did not take advantage of any structural periodicity.

In the transfer matrix method, the generalized displacements and forces at the left-hand end of one periodic element $A$ are combined into the state vector $\{Z\}_A = [q_s, Q_s]^T$. $[q_s]$ represents the displacements and/or rotations at the end and $[Q_s]$ represents the corresponding forces and/or moments. The state vector at the left-hand end of the next periodic element $B$ is $\{Z\}_B = [q_s, Q_s]^T$. 
These two state vectors are related through the “period transfer matrix” $[T]_{A,B}$ such that

$$\{Z\}_B = [T]_{A,B} \{Z\}_A.$$ \hfill (9a)

$[T]_{A,B}$ can be found by appropriate transformation and reorganization of the receptance equation relating the $q_i$’s and $Q_x$’s or by other means. Its form is well known for beam, bar and shaft elements. When a free wave travels along the periodic system with the propagation constant $\mu_j$, the state vectors are related by Floquet’s principle, so

$$\{Z\}_B = \lambda_j \{Z\}_A,$$

where $\lambda_j = \exp(\mu_j)$. Hence

$$[T]_{A,B} \{Z\}_A = \lambda_j \{Z\}_A$$ \hfill (9b)

and $\lambda_j$ is an eigenvalue of $[T]_{A,B}$. Transfer matrices fall into the category of “symplectic matrices”, and therefore have a number of very useful properties which have been exploited in modern control theory and periodic structure analyses. Lin and McDaniel [24] showed how the eigenvalues and corresponding eigenvectors can be used to enhance the computational efficiency when predicting the forced harmonic responses of long “compound” beam-type periodic structures and the internal noise levels of a quasi-one-dimensional stiffened cylinder excited by a random pressure field. The useful properties of transfer matrices were also recognized and used by Sen Gupta at Southampton [14]. Having demonstrated that the propagation constants can be found from the eigenvalues of the transfer matrix, he continued his analysis to prove the orthogonality of the different wave motions which can occur at the same frequency. From this he derived a “Rayleigh quotient” for the propagation frequency of a wave of given propagation constant and known waveform. Using the quotient with some reasonable approximate wave modes and phase constants, he found good approximate values for the frequencies of the waves.

In his Ph.D. thesis in 1974, De Espindola [28] used transfer matrices to study both free and forced wave propagation along a cylinder with either periodic circumferential stiffening or longitudinal stiffening. In both cases the shell displacements varied...
sinusoidally in one direction or the other so the structures were quasi-one-dimensional and
transfer matrix methods were applicable. Much effort was required to overcome numerical
ill-conditioning and inefficiency in the computational processes. De Espindola followed
Henderson and McDaniel [26] and found the field transfer matrix for the cylinder element
by using the constituent idempotents of the fundamental state matrix of the element.
He also showed how the eigenvalue problem can be halved when it is known that the
$2N_x$ eigenvalues always exist in reciprocal pairs. From the characteristic equation of the
period transfer matrix (of order $2N_x$) he established a polynomial equation for $\cosh \mu$ of
order $N_x$.

Attention must be drawn at this stage to the extensive work on periodic structures by
Williams and his group at Cardiff. In a recent publication, Zhong and Williams [29] utilized
the symplectic property of period transfer matrices to develop more efficient and accurate
computational procedures. They found that the matrix eigenvalue problem can be set up
in terms of $\cosh \mu$ rather than $\exp(\mu)$, so only $N_x$ eigenvalues need be (and can be)
computed by computer matrix-eigenvalue routines. This contrasts with De Espindola’s
relatively tedious method of setting up and solving the characteristic equation. Zhong
and Williams also considered wave-scattering when a single free wave in a general
one-dimensional system impinges on an arbitrary boundary and gets reflected and
transmitted into other waves. (See also reference [19]). They cited other useful references
which apply transfer matrix methods to periodic structures.

6. THE METHOD OF SPACE-HARMONICS

In the late 1960s, investigations began at Southampton into the sound radiated from
the surfaces of periodic structures and the effects of fluid loading (acoustic damping) on
the forced harmonic response. These influences are analyzed most conveniently if the
surface motion can be decomposed into spatially harmonic components and the method
of “space-harmonics” was developed for this purpose. The complex free wave motions in
a beam which had been computed by receptance methods were identified as groups of
sinusoidal waves with all wavenumbers $(\omega/n\pi + n\pi)/L_x$ $(n$ is any positive or negative integer).
In our earliest studies [30] we investigated the amplitudes of the components in these
groups, half of which travelled in the positive $x$ direction and half in the negative $x$
direction. These studies threw much light upon the mechanism of “coincidence-type”
excitation of periodic structures by external convected pressure fields.

When the harmonic pressure field, $p_0 \exp(i\omega t - k_x x)$ convects along a periodic beam,
each periodic element undergoes the same flexural displacement apart from the phase
difference of $k_x L_x = \omega_0$ between adjacent elements imposed by the pressure field. Due to the
ambiguity of the phase angle (it could be $\omega_0 \pm 2n\pi$), the transverse displacement at any point
$x$ along the whole infinite beam can be represented by the space harmonic series

$$w(x, t) = \sum_{n=-\infty}^{\infty} A_n \exp i[\omega t - (\omega_0 + 2n\pi) x/L_x].$$ (10)

It was called a “space-harmonic series” to distinguish it from a normal Fourier series,
which does not have the $\omega_0$ term added to the $2n\pi$. The complex amplitudes $A_n$ generated
by the pressure field are given by an infinite set of equations which were set up by a
variational method in the process of which only one element of the periodic structure
needed to be considered. Account was readily taken of the potential and kinetic energies
of the beam element and its elastic supports and also of the work done by the excitation
pressure \( p_0 \) and the reaction pressure on the element from the adjacent fluid medium. The equations for the \( A_n \)'s were found in the familiar form

\[
[[K] + io[B] - \omega^2[M]] \{A_n\} = \{P\}, \quad (11)
\]

where \([K] = [K(\varepsilon_n, n, EI, L_x, \kappa_x, \kappa_y)]\) is a full, symmetric stiffness matrix and is a function of the enclosed parameters. \( \kappa_x \) and \( \kappa_y \) characterize the rotational and transverse stiffnesses of the periodic supports respectively. \([B] = [B(\varepsilon_n, n, \rho_m, c_m, L_x)]\) is a diagonal damping matrix, which also depends on the density and speed of sound in the fluid medium, \( \rho_m, c_m \). \([M]\) is a diagonal mass matrix and \( \{P\} = [0, 0, \ldots, 0, p_0, 0, 0, \ldots]^T \) is a single-element force vector, the non-zero element belonging to the equation for \( n=0 \).

Direct numerical solution of the infinite set of equations (11) required the set to be truncated and the number of terms required was investigated by Mead and Pujara [31]. While the response level could be determined quite accurately with only five terms, and very accurately with 11, the sound radiated by the beam or plate often requires many more terms before it converges satisfactorily. The computed response of the periodic beam to a harmonic pressure field was then used to find the r.m.s. response to a homogeneous random convected pressure field with a known wavenumber–frequency spectrum [32]. In his Ph.D. thesis, Pujara [32] extended the method to deal with the two-dimensional orthogonally stiffened plate under a convected random pressure field. For this case the space harmonic series was doubled into the form

\[
w(x, t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \exp [i(\omega t - (\varepsilon_{nx} + 2\pi n)x/L_x - (\varepsilon_{ny} + 2\pi m)y/L_y)] \quad (12)
\]

where \( \varepsilon_{nx} = k_{px}/L_x, \varepsilon_{ny} = k_{py}/L_y \) and \( k_{px}, k_{py} \) are the wavenumbers of the pressure field in the two directions. Equations for the \( A_{nm} \)'s were set up and solved in the same way as before. Pujara included the effects of fluid loading (air) and proceeded to compute the sound power radiated from the plate.

Space-harmonic analysis at Southampton was continued by Mace, who considered a two-dimensional plate periodically stiffened in one direction and excited by a convected pressure field of plane harmonic waves [33]. Using a Fourier transform method he obtained an expression for the plate displacement in the same space-harmonic form as equation (10). His analysis led to explicit expressions for the coefficients \( A_n \) when the adjacent fluid exerted no reaction pressure and this eliminated the need to solve simultaneous equations for the \( A_n \)'s. He proceeded to investigate the sound radiated by the same plate excited either by a sinusoidal line force parallel to the stiffeners or by a point force at an arbitrary location [34]. While the responses of these plates can be found analytically when there is no fluid reaction, numerical integration must be used when this reaction is present.

Fluid loading adds effective mass and damping to a vibrating plate so the frequencies of free wave motion and the boundaries of the propagation zones are changed. Mace investigated this in 1980 [34], again for the plate which was periodically stiffened in just one direction. He observed that even within the propagation zones any free wave motion must now decay as it propagates, since energy is lost from the plate by acoustic radiation into the medium. This feature was investigated further by Mead in 1990 [35], using the space-harmonic series in conjunction with the concept of “phased array receptance functions” for periodic structures [36]. Assigning the frequency, he developed an iterative process of finding the complex propagation constants which satisfied a particular set of equations. Although it was possible in this way to find the constants for a lightly loaded periodic plate (e.g., a steel plate loaded by air) only limited success was achieved when the plate was much more heavily loaded by water.
The space-harmonic series has also been used by other investigators (not at Southampton), notably in 1985 by Hodges et al. [37] and in 1994 by Williams et al. [38]. Hodges used it to find the low order natural frequencies and modes of circumferentially stiffened cylindrical shells. A circumferential mode order and an $x$-wise propagation constant $\varepsilon_x$ were assigned, a truncated set of equations such as equations (11) was set up and the frequencies of propagation were found as the eigenvalues. Hodges included the effects of cross-sectional distortion of the circumferential stiffeners† and showed that the number and width of the propagation zones can be dramatically changed by its inclusion. Natural frequencies computed by this space-harmonic method agreed well with experimentally measured values [41].

Williams et al. [38] used the space-harmonic approach within a powerful general purpose computer program to calculate the natural frequencies of very large structures. Values of $\varepsilon_x$ were first assigned, but were restricted to rational fractions of $\pi$. This implies that the corresponding wave motions repeat themselves over an integral number of periodic elements, over which distance the space-harmonic series became a general Fourier series and allowed Williams' computer programme to be used for the efficient computation of frequencies of large periodic structures.

7. THEOREMS RELATING TO WAVE MOTION IN PERIODIC STRUCTURES

Over the period 1973–1975, a number of papers from Southampton presented some vibrational principles and general properties relating to the wave motion. These were required as a basis for finding the natural frequencies of finite periodic structures and for establishing approximate methods of calculating free and forced wave motion in periodic structures which were genuinely two-dimensional rather than quasi-one-dimensional. In his Ph.D. thesis of 1973 [42] (see also reference [43]), Abrahamson used Hamilton’s variational principle to show that the frequency of a freely propagating wave in a periodic structure satisfies a stationary property with respect to small variations in the waveform. From this he established the following (not unexpected) Rayleigh quotient for the frequency of a single, albeit approximate complex wave mode $w(x)$ of given phase constant $\varepsilon_x$:

$$\omega^2 = \int_0^L EI|w'/(v^2)|^2 \, dx \int_0^L \rho A|w'|^2 \, dx.$$  \hspace{1cm} (13)

This differs from the normal Rayleigh quotient by virtue of the modulus signs within the integrals. The approximate wave-mode must satisfy the geometric wave-boundary conditions.† Abrahamson then used the stationary property to develop a “Rayleigh–Ritz” method for finding the propagation frequencies of a series of approximate wave modes, each of which had to satisfy the geometric wave-boundary conditions for a given value of $\varepsilon_x$.

Also in 1973, Mead [45] presented his general theory of harmonic wave propagation in multi-coupled periodic systems of one or two dimensions. It was shown that the number of different waves (propagating or attenuating) which can travel in a one-dimensional system at any frequency is twice the number of coupling co-ordinates, $N_x$, which exist at the periodic junction at which this number is smallest. (For a simply supported periodic beam, there is one coupling co-ordinate at a simple support, but there are two at the centre.

† The important effects of this on natural frequencies had previously been investigated at Southampton by Beresford [39] and Giannopoulos [40].‡ If it also satisfies the natural wave-boundary conditions, a more accurate value is obtained for $\omega$ [44].
of a bay. The smallest number is \( N_x = 1 \), so just two different waves can exist.) The energy transported by the different types of wave was also considered in reference [45].

The most important advance in reference [45] was the formulation of generalized equations of motion for a periodic element within a multi-coupled periodic system through which a wave is propagating. The motion in the element was quantified by a set of generalized co-ordinates, each of which was associated with a real mode as distinct from Abrahamson’s complex modes. Lagrange’s equations were used to set up an initial set of real generalized equations of motion. The wave-boundary conditions were then applied to these equations,† which were thereby reduced in number. The coefficients of the reduced set contained the propagation constant(s) assigned in the boundary conditions. As in Abrahamson’s method, the equations are solved for the eigenvalue propagation frequencies corresponding to the assigned values of \( \varepsilon \), for a one-dimensional system, or of \( \varepsilon_x \) and \( \varepsilon_y \), for a two-dimensional system. Alternatively, they can be solved for the eigenvalue-propagation constants for a one-dimensional system (i.e., for the \( \lambda_x \)’s) for a given frequency. Further details about this method will be presented in a later section.

Also introduced in 1973 [45] was the concept of “forced damped normal wave motion” in hysteretically damped periodic structures. Such waves form the constituent motions of “damped-forced normal modes” of heavily damped, periodically stiffened finite plates. They are forced by distributed harmonic pressures which are in phase with the local plate velocity but are proportional to the local inertia force. The constant of proportionality was identified in reference [45] with the loss factor of the wave which (being forced) does not decay as it propagates. The concept of these waves was used in 1976 [46] to find the loss factors and propagation frequencies of waves in quasi-one-dimensional periodic sandwich plates.

In 1975 [18, 19], the concept of characteristic waves and wave vectors for periodic structures was developed. Wave \( j \) in the set of \( 2N_x \) different waves of a one-dimensional structure is associated with its complex eigenvector \( \{ A \} \), obtained from equations such as equations (8b). From the \( \{ A \} \)’s one can find a complete set of \( 2N_x \) normalized complex characteristic wave functions \( W_j(x) \), each of which describes the displacements at any point within the periodic element due to one of the waves. By introducing the generalized wave co-ordinates \( \Psi_j \), one can express the total free wave motion when all the characteristic waves are presented by

\[
 w(x) = \sum_{j=1}^{2N_x} \Psi_j W_j(x) = [ W_j(x) ] [ \Psi ] . \tag{14} 
\]

Each characteristic wave propagates or decays along the system according to its own propagation constant, quite independently of the other waves. If equation (14) represents the displacement in bay 0, the displacement in another bay, \( k \) bays to the right, is given by

\[
 w(x_k) = \sum_{j=1}^{2N_x} \Psi_j \exp(k\mu_j) W_j(x_k) \tag{15} 
\]

(\( x \) and \( x_k \) are measured from the left-hand ends of their respective bays). This equation allows very efficient computation of response at points in a periodic structure very remote from sources of excitation. It was used in reference [47] to investigate the decay of forced

† This process is analogous to the extended Rayleigh–Ritz method of setting up equations for approximate natural frequencies of beam-type structures. It also corresponds to the process in finite element calculations when boundary constraints are applied to the equations after they have been set up.
harmonic motion along a damped beam in which flexural and longitudinal wave motions were coupled. In a particular case, the spatial decay rate of the total motion appeared to have two different values depending on the distance from the single harmonic source. This followed from the existence in the total motion of two different characteristic waves decaying at their two different rates.

Equation (15) was also used in reference [19] to show that, if just one positive-going wave impinges upon a boundary, each of the \( N \) negative-going waves can be reflected back into the system. Furthermore, if the periodic system contains a discontinuity such as one non-periodic element upon which a single positive-going wave impinges, then both the transmitted and reflected motions may contain \( N \) different characteristic waves.

Also presented in references [18] and [19] were formal proofs of the relationship between the bounding frequencies of propagation zones and the natural frequencies of an isolated element of the periodic system. If the periodic elements are symmetric about their spanwise centres, the bounding frequencies are the same as the natural frequencies of an isolated element with its ends either “fixed” or “free” in a carefully defined sense. This is true for both mono-coupled and multi-coupled systems. If the periodic element is asymmetric, its own natural frequencies inevitably occur just outside the propagation zones of the periodic system.

It was formally proved in references [18] and [19] that Sen Gupta’s method [16] of finding the natural frequencies of finite periodic structures via the propagation (phase) constant can be generalized (in principle, at least) to apply to structures which have arbitrary conservative boundary conditions and which are mono-coupled or multi-coupled, one-dimensional or rectangular-two-dimensional. The phase closure principle still applies to such structures and leads to an equation for the phase constants at which natural frequencies occur. Due allowance must now be made for the changes of phase of a characteristic wave as it is reflected from the extreme boundaries, and this may lead to considerable practical difficulties in finding the frequencies, most of which will still fall in propagation zones. It was also shown [19] that any evanescent wave which is generated in the reflection process must undergo an overall attenuation factor of unity as it, too, makes its own complete circuit of the finite system.

More recent and significant contributions to fundamental periodic structure theory have been made by Langley [48] (formerly at Cranfield, now at Southampton) and by Zhu [49] (of Beijing University of Aeronautics and Astronautics, and a former Senior Visiting Research Fellow at Southampton). They have presented variational principles which apply particularly to wave motion in periodic structures. Using these, one can set up equations for the forced motion and the propagation frequencies in ways which differ from that of reference [45] and which have some advantages over it. Space is insufficient to present further details here.

8. APPLICATIONS OF ENERGY METHODS TO WAVE MOTION STUDIES

Flat plates and cylinders with periodic stiffening in both \( x \) and \( y \) directions are not, in general, reducible to quasi-one-dimensional structures. Closed form solutions do not exist for their characteristic free waves or for their forced responses, so these can only be studied by approximate energy methods such as those described in section 7.

Abrahamson applied his “Rayleigh–Ritz” method to find the frequencies of wave motion in periodic beams and rib–skin structures [42]. Each approximate complex mode had to satisfy the geometric wave-boundary conditions corresponding to the phase constant \( \epsilon_r \). For a simply supported beam the mode was expressed by \( w(x) = q_f(x) + i q_f(x) \), where both \( f_i(x) \) and \( f_r(x) \) were real, approximate and suitably
chosen modes which satisfied the geometric boundary conditions of a single isolated element of the structure. The proportion \( q_i : q_j \) was adjusted to satisfy the wave-boundary condition \( w'(L) = \exp(-i\omega t)w(0) \). The whole process of finding suitable complex modes and then of computing the corresponding complex terms in the stiffness and inertia matrices was clearly cumbersome. Nevertheless, suitably chosen modes led to computed propagation frequencies which compared well with values obtained from exact solution.

This same process of using complex wave modes with assigned phase constants was also used by Mead and Mallik in 1976 [44] to compute the approximate response of a periodic beam subjected to a convected harmonic pressure field. In 1979, Mead and Parthan [50] considered plane propagating wave motion and natural frequencies in two-dimensional flat plates on periodic rectangular grids of simple supports. The approximate modes used for the periodic plate elements were complex combinations and products of either simple polynomial functions or of the natural modes of simply supported and fully fixed beams. Each combination had to satisfy the geometric wave boundary conditions corresponding to the pair of phase constants \( \varepsilon_x \) and \( \varepsilon_y \) which defined the plane wave motion.† These constants were arbitrarily chosen within the admissible ranges \( 0 \) to \( \pm \pi \) and the eigenvalue propagation frequencies were computed.

When these frequencies are plotted in a three-dimensional form against \( \varepsilon_x \), \( \varepsilon_y \), the “phase constant surfaces” of Figure 8(a) are obtained. If the periodic plate element is doubly symmetrical and the phase constant surfaces are plotted over the whole range \( -\pi < \varepsilon_x < \pi \), \( -\pi < \varepsilon_y < \pi \), they have four-fold symmetry about the frequency axis. This is illustrated in Figure 8(b) for the lowest surface. Finite periodic plates with \( n_x \) symmetrical elements in the \( x \)-direction and \( n_y \) in the \( y \)-direction have natural frequencies where \( \varepsilon_x = j_x \pi / n_x \) and \( \varepsilon_y = j_y \pi / n_y \) (\( j_x \) and \( j_y \) are integers between 0 and \( n_x \) and 0 and \( n_y \) respectively). Parthan used this in conjunction with the above method to find very accurate values of the natural frequencies of simply supported periodic plates. The Rayleigh–Ritz method was also extended to deal with the forced harmonic motion in the plate due to a random frozen-convected plate pressure field. It was shown that there are “preferred” directions in which the pressure field should convect in order to maximize or minimize the plate response. This leads to a simple criterion for the design of the plate if its response is to be minimized.

The approximate energy method which has received most attention is that based on the equations formulated in reference [45] and introduced briefly in section 7. The motion of one element of the two-dimensional periodic structure was quantified by a finite number \( N_{TOT} \) of generalized displacement co-ordinates \( \{q_r \} \). \( N_r \) of these were along each of the left- and right-hand edges of the element and \( N_i \) were along each of the top and bottom edges, \( N_i \) were inside the element and \( 4N_c \) were allowed for the four corners, \( N_c \) per corner (see Figure 9). A real displacement mode was associated with each co-ordinate and a generalized force or moment with each boundary co-ordinate. The matrix displacement method was used to set up the generalized equations of free harmonic motion in terms of these co-ordinates; these have the usual form

\[
[D(\omega)] \{q_r \} = [K - \omega^2 M] \{q_r \} = \{Q_r \}. \tag{16}
\]

\([D(\omega)]\) is a square dynamic stiffness matrix of order \( N_{TOT} \times N_{TOT} \). When the motion is free, the \( N_i \) generalized forces \( Q_i \) are zero and the internal co-ordinates \( q_i \) and their corresponding equations can be eliminated from the equations. The generalized forces

† A “plane wave” in a two-dimensional periodic structure is defined as one which has the pair of propagation constants \( \varepsilon_x \) and \( \varepsilon_y \) in the \( x \)- and \( y \)-directions. If the periodic lengths in these directions are \( L_x \) and \( L_y \), the direction of the wave motion relative to the \( x \)-axis is \( \theta = \tan^{-1} (\varepsilon_y / \varepsilon_x) (L_y / L_x) \). Langley [51] has discussed the direction of energy flow in such waves.
Figure 8. (a) Phase constant surfaces for a two-dimensional periodic plate; (b) the extended phase constant surface of the lowest surface, showing its four-fold symmetry.
corresponding to the edge and corner co-ordinates are those which are derived from the internal forces and moments of interaction between the adjacent periodic elements.

As stated before, the wave-boundary conditions in this method are applied to equations (16). Equivalent plane wave motion can be assumed to exist with propagation constants $\mu_x$ and $\mu_y$ (or $\epsilon_x$ and $\epsilon_y$) in the $x$- and $y$-directions. With $\exp(\mu_x) = \lambda_x$ and $\exp(\mu_y) = \lambda_y$, Floquet's principle relates the left and right co-ordinates by $\{q_R\} = \lambda_x \{q_L\}$, the top and bottom co-ordinates by $\{q_T\} = \lambda_y \{q_B\}$ and the corresponding generalized forces by $\{Q_R\} = -\lambda_x \{Q_L\}$ and $\{Q_T\} = -\lambda_y \{Q_B\}$. The corner co-ordinates and forces, $q_{LB}$, $q_{LT}$, $Q_{LB}$, $Q_{LT}$, etc., have similar relationships. Altogether, this allows equations (16) to be reduced in number from $N_{TOT}$ to $N_{RED} = (N_x + N_y + N_c)$ and to be expressed in terms of the $\{q_L\}$, $\{q_B\}$ and $\{q_{LB}\}$ co-ordinates only. The equations of free wave motion now take the form

$$[[K'(\lambda_x, \lambda_y)] - \omega^2[M'(\lambda_x, \lambda_y)]]\{q_{L,B,LB}\} = 0,$$

where $\{q_{L,B,LB}\} = [q_L, q_B, q_{LB}]^T$. The elements of the matrices $K'$ and $M'$ are functions of the propagation constants as well as of the stiffness and mass of the periodic element. If $\epsilon_x$ and $\epsilon_y$ (and hence $\lambda_x$ and $\lambda_y$) are assigned, these matrices are Hermitian and $N_{RED}$ real frequencies $\omega$ can be found as the eigenvalues of the equation. Equation (17) and its forced vibration counterpart have been used extensively in periodic structure studies both at Southampton and elsewhere.

Orris and Petyt at Southampton [52] were the first to use these equations in conjunction with the "h-version" of the finite element method (FEM). Considering one-dimensional periodic beams and rib–skin structures, they split up each periodic element into a finite number of sub-elements (up to eight in their case). The assumed modes for each sub-element were the standard cubic functions. Mass and stiffness matrices for equations (17) were compared for given values of $\epsilon_x$ after which the eigenvalue propagation frequencies were computed. The frequencies obtained with 2, 4 and 8 sub-elements per periodic element were compared with the values found by Sen Gupta et al. by exact methods from closed form solutions. In the first two or three propagation bands, the use of four sub-elements gave frequencies of very good accuracy and eight sub-elements gave excellent accuracy. Orris and Petyt [53] proceeded to show how the FEM can be used to compute the response of infinite periodic beams and rib–skin structures to convected random pressure fields.

Under Petyt's direction, Abdel-Rahman extended the use of the FEM to periodic structures, but regrettably never published any part of his excellent Ph.D. thesis of 1980. He considered beams on periodic elastic supports, flat plates with periodic flexible stiffeners
in orthogonal or angled arrays and three-dimensional periodic beam systems. The $h$-version of the FEM was used to split up the periodic elements and their stiffeners into manageable numbers of sub-elements and the usual plate or beam FE co-ordinates constituted the co-ordinates $\{q_n\}$. Results were presented from extensive calculations for both free wave propagation constants and forced response levels generated by convected pressure fields. Polar contour plots of the phase constants versus frequency were drawn for a periodic plate and these show how the frequency for a given phase constant varies with direction of the plane-wave propagation over the plate. An example of such a plot in Figure 10 clearly shows those directions in which the propagation zones are widest or narrowest. The discontinuities in the contours correspond to the discontinuities identified by Brillouin in his discussion on the reciprocal lattices of two-dimensional periodic systems [1].

Abdel-Rahman also investigated the more difficult problem of finding the attenuation constants of decaying wave motion in preferred directions over a periodic plate. Equations (17) can be rearranged into a form which yields the $\lambda$’s as eigenvalues after $\omega$ is assigned [45]. If the $\lambda$’s correspond to real $\mu$’s, a polynomial eigenvalue problem has to be solved. The eigenvalues can only be computed in a reasonable time if one $\mu$ (say $\mu_x$) is equal to the other ($\mu_y$) times a very low integer. The higher is the integer, the higher is the order of the polynomial problem. Abdel-Rahman only had time to find the $\mu$’s for attenuating waves which travelled across a periodic plate with square elements in directions 0°, 45° and 90° to the $x$-axis. He considered the responses of finite and infinite periodic systems to convected pressure fields and demonstrated the great value of periodic structure.

![Figure 10. Phase constant contours of the propagation surfaces for a two-dimensional flat plate on simple supports; rectangular cells with $L_x/L_y=0.5$; first two propagation bands only; $\theta =$ direction of the "plane-wave" motion across the plate; $\Omega =$ non-dimensional frequency.](image-url)
theory—that the response and natural frequencies of a finite periodic system can be estimated by appropriate consideration of just one of its elements. Periodic structure theory was therefore vindicated as a means of drastically reducing the FE computation time required to compute frequencies and vibration levels of large multi-bay periodic structures of finite extent.

Equation (17) was used by Mead’s group in the late 1980s to study propagating wave motion in orthogonally stiffened periodic flat plates and cylindrical shells [22, 23, 55–58]. The hierarchical finite method (“HFEM”—the “p-version” of FEM) was used in which the periodic plate element and its surrounding stiffeners were treated as single elements and were not subdivided. The variations with \(x\) and \(\theta\) of the assumed transverse displacements within the curved periodic element were represented by the four Hermite cubic functions of standard beam elements, together with a hierarchy of other polynomial functions which have zero displacement and first derivative at the plate or beam boundaries. The in-plane displacements were represented by simple linear functions and another hierarchy of polynomial functions. The total displacement of the plate at any point was represented by a double series having the general form

\[
 u(x, \theta, t), v(x, \theta, t) \quad \text{or} \quad w(x, \theta, t) = \sum_{i=1}^{t} \sum_{j=1}^{l} q_{ij}(t) F_i(x) G_j(\theta). \quad (18)
\]

Terms in this which involve products of the hierarchical functions correspond to the internal co-ordinates, \(q_i\). The other terms correspond to the external co-ordinates \(q_L, q_R, q_B\) and \(q_T\). The actual hierarchy of functions used was Rodrigues’ form of the Legendre orthogonal polynomials, and these led to the vanishing of many of the off-diagonal terms in the stiffness matrices. Symbolic computer processing was essential for the reliable evaluation of the numerous integrals (over 1000) of the products of the functions and their derivatives required in the stiffness and mass matrices.

Extensive calculations were undertaken to find the propagation frequencies of waves around and along the cylinder with pre-assigned values of \(o_x\) and \(o_u\) between 0 and \(\pi\). Over 20 functions described the \(x\)-wise and \(\theta\)-wise variation of displacements and these led to matrix eigenvalue equations for the frequency of order 400 × 440 or more. Phase constant surfaces were computed for a number of different cylinder–stiffener configurations. Computer plots of some of the corresponding wave modes are shown in Figure 11. The bending and torsional rigidities of the stiffeners on the cylinder were taken fully into account, but not the effect of distortion of the stiffener cross-section. This can be allowed for by introducing further internal degrees of freedom into the stiffener motions.

The natural modes and frequencies of finite stiffened cylinders can be found as for a two-dimensional periodic structure by applying the phase closure principle. If the stiffened cylinder has \(n_x\) periodic bays along its length and \(n_u\) around its circumference, natural modes exist at frequencies at which \(o_x = j_x \pi / n_x\) and \(o_u = 2j_u \pi / n_u\) (\(j_x\) and \(j_u\) are integers between 0 and \(n_x\) and 0 and \(n_u\) respectively). This condition for \(o_u\) is discussed in greater detail in reference [22].

More recently, Bardell and Langley [58] have used the HFEM to investigate wave propagation in flat plates resting on oblique arrays of periodic line-simple supports. Plane waves can propagate in these structures with phase constants \(\varepsilon_x\) and \(\varepsilon_y\) across the opposite sides of the “skewed” periodic plate elements. Frequencies for given pairs of \(\varepsilon_x\) and \(\varepsilon_y\) were computed and led to phase constant surfaces such as that in Figure 12. These do not have the same four-fold symmetry as the surfaces for plates and cylinders with rectangular...
stiffening arrays, and this makes it impossible to use Sen Gupta's method to determine the natural frequencies of a finite skewed periodic plate. It is probable that the natural modes can no longer be regarded as superpositions of plane wave motions across the periodic plate surface.

Again using the HFEM, Bardell et al. [59] investigated one-dimensional wave motion along beams with asymmetric periodic elements. An off-centre mass in a uniform element reduces the widths of the frequency propagation zones. The bounding frequencies of the propagation zones no longer coincide with the natural frequencies of a single beam element with free or fixed end conditions and these natural frequencies now occur inside attenuation zones. Asymmetry caused by an extra off-centre support added within each beam element makes the low frequency propagation zones very narrow. This can reduce the transmission of wave motion along a damped beam away from a point source, but does not reduce the response due to a random distributed pressure field.
9 THE METHOD OF PHASED ARRAYS OF FORCES AND MOMENTS

This is an exact (closed form) method of studying the free and forced wave motion in quasi-one-dimensional periodic structures and was first used at Southampton in the 1980s [36]. When a single characteristic wave travels through a periodically supported continuous structure, the supports react on the structure with forces and/or moments which are identical apart from a fixed phase angle \( \phi \) or an attenuation constant \( \mu \) from one support to the next. The displacement at any point in the infinite continuous structure due to a single harmonic force \( P_0 e^{i\omega t} \) at \( x=0 \) can be expressed in the general form

\[
w_r(x, t) = P_0 e^{i\omega t} \sum_{n=1}^{N_x} a_n e^{-\beta_n x}
\]

for points to the right of the force and

\[
w_l(x, t) = P_0 e^{i\omega t} \sum_{n=1}^{N_x} a_n e^{+\beta_n x}
\]

for points to the left. The \( \beta \)'s are the wavenumbers of free wave motion in the structure. \( N_x \) is equal to one half the order of the governing wave equation for the continuous structure, and is the same as the number of coupling co-ordinates in the periodic structure when there is no rigid constraint at the junction. The \( a_n \)'s are found by standard methods of flexural wave mechanics.

Now suppose an infinite set of such forces with the attenuation constant \( \mu \) (or the phase constant \( \phi \)) from one force to the next acts on the structure at equal intervals \( L_x \); i.e., the set constitutes a "phased array". The displacement of any force location is found from an infinite sum of the above displacements, and was shown in reference [36] to be

\[
w_{\text{tot}}(0, t) = -P_0 e^{i\omega t} \sum_{n=1}^{N_x} a_n \frac{\sinh \beta_n L_x}{\cosh \mu - \cosh \beta_n L_x} = x_{pp} P_0 e^{i\omega t}.
\]  \hspace{1cm} (19)

Expressions of similar form were found for the displacement due to a phased array of harmonic moments and for the gradients (slopes, \( w'(x, t) \)) due to phased arrays of forces.
or moments. \(a_{PP}\) is a “phased array receptance function” and the whole set of such functions are basic ingredients used in the method of phased arrays for studying wave motion in continuous periodic structures.

The simplest such structure is a beam on simple supports which can exert no moments on the beam. When a free characteristic wave travels through the beam, the only forces on the beam are the support reactions (forces) and these create zero displacements at the supports. Hence \(a_{PP}=0\). For a uniform Euler–Bernoulli beam, \(\beta_2 = i\beta_1 = \beta\) and \(a_2 = ia_1\).

Substitution of these into equation (19) leads (after manipulation) to

\[
\cosh \mu = (\cos \beta L, \sin \beta L, -\sin \beta L, \cosh \beta L)/(\sinh \beta L, -\sin \beta L),
\]

which is identical (as it should be) to equation (1).

This example demonstrates the simple way in which an equation for the propagation constants can be obtained by the method of phased array receptance functions. Further examples are given in reference [36] in which wave propagation in periodic Timoshenko beams is considered (i.e., allowing for shear deformation in the beam) and also in a quasi-one-dimensional stiffened plate with stiffeners which have both torsional and flexural flexibilities.

A comprehensive study of the various phased array receptance functions was presented in 1989 by Yaman, in his Ph.D. thesis [60]. He considered Euler–Bernoulli beams, quasi-one-dimensional Kirchhoff plates and three-layered damped sandwich beams and plates and found propagation constants for these when periodically stiffened. The use of the functions was particularly helpful when the responses of periodic structures to point forces or line harmonic loads were calculated [61]. Yaman found space-averaged response levels for a six-bay, three-layered damped sandwich plate which compared well with experimentally measured values [62].

10. DISORDERED PERIODIC STRUCTURES

A single disorder exists in a periodic structure when just one of its elements differs from the rest by virtue of its length, stiffness, mass, etc. In his Ph.D. thesis [63], Bansal examined the effect of a single length-disorder on the propagation wave motion along an infinite simply supported periodic beam. By means of receptance methods he computed the motion transmitted across and reflected from the disorder when a single characteristic wave impinged on it from one side. The disorder was found to resonate with high response levels in the frequency attenuation zones of the periodic structure [64]. The same structure excited by a convected harmonic pressure field had peak responses at these resonance frequencies and also at the wave-coincidence frequencies of the free wave motion and the pressure field [65].

The only multiple disorders studied at Southampton before 1990 were “periodic disorders” which occur regularly throughout the whole infinite structure. Sen Gupta [14] presented propagation constants for a periodic beam on supports of infinite transverse stiffness and with finite rotational constraints of stiffness \(K_r\), increased at every sixth support to \(2K_r\). The effect was to split up the propagation zones into six small “sub-zones” with five attenuation zones between them. The total frequency range in which free waves could propagate was therefore greatly reduced by such disordering.

Bansal [63, 66] considered multiple periodic length-disorders in simply supported beams. He used receptance methods to set up equations for the propagation constants and presented results for beams in which all successive four-bay sections were identically disordered. Once again, the propagation zones were found to split into \(N\) sub-zones, \(N\)
being the number of bays over which the disorder was repeated \((N=4\) in Bansal’s case). Due to the disordering, the attenuation constants of damped beams were increased in the former propagation zones but were decreased in the former attenuation zones.

The response of a periodically disordered beam to a convected harmonic pressure field was considered by Bansal in reference [67]. The coincidence phenomenon was found to occur at least once in each of the propagation-frequency sub-zones, so a beam with periodic disorders extending over \(N\) bays had \(N\) times as many coincidence peaks in the response spectrum as the ordered beam. Not all of these peaks were really significant, but their combined effects increased the overall response level. Using a space-harmonic analysis, Bansal proceeded to compute the sound power radiated by a periodically disordered beam excited by a random frozen-convected pressure field. Numerous different configurations were examined with randomly selected length-disorders. The radiated sound power was found to increase with an increase in the degree of disorder [63].

Studies of the effects of disorders in periodic systems are continuing at Southampton under Langley [68, 69]. Whereas in the early work deterministic periodic disorders were considered, in this later work non-periodic, truly random disorders are being investigated and this is adding to the important work of Pierre and colleagues, Cai and Lin, Kissel and others over the past decade.

11. EXPERIMENTAL MEASUREMENTS OF PROPAGATION CONSTANTS

Reference has already been made to experimental work on periodic structures in which harmonic response levels were measured and found to agree well with theoretical values [21, 62]. Only one attempt has been made at Southampton (or elsewhere, to the author’s knowledge) to measure the actual propagation constants. This was undertaken in the mid–1970s by Ohlrich (working with F. J. Fahy) who studied a simple model consisting of a long beam across which a set of shorter beams was periodically and symmetrically attached [70, 71]. Both longitudinal and flexural wave motion could occur in this model, which was therefore a tri-coupled system. Ohlrich deduced the propagation constants from measurements of transfer receptances made at the periodic junctions on a finite eight-bay model. This was driven at one end with a rapidly swept sinusoidal force and embedded at the other end in a low reflective termination. If only one wave is propagating, the propagation constant is given by the natural logarithm of the ratio of these receptances at a pair of adjacent junctions. More accurate results were obtained from an appropriate average value of this logarithm over several adjacent and non-adjacent junctions. Ohlrich exercised great care and ingenuity in ensuring that only one wave-type was excited and he thereby obtained results which agreed well (some extremely well) with theoretical values. The swept frequency excitation and subsequent data analysis meant that phase constant curves were obtained directly in the unique form of Figure 1(a).

12. CONCLUSIONS

As a result of the work of the past 30 years, characteristic freely propagating wave motion can now be readily computed for one-, two- and three-dimensional continuous periodic structures. Although most of this work has applied to uniform beams, plates and shells, the recent energy methods of analysis can be extended to deal with non-uniform periodic elements. The forced wave motion generated by single-point harmonic forces or plane harmonic pressure waves on infinite periodic structural surfaces can also be readily computed. Appropriate Fourier transformation in the time and/or space domains of these
responses enables one to compute the responses of periodic structures to both fixed and moving point harmonic loads, point impulsive loads and random pressure fields. This has yet to be undertaken for two-dimensional periodic structures under single-point harmonic or impulsive loading. It will be useful to study the directivity of the energy flux in this two-dimensional case.

The natural frequencies, modes and forced responses of finite periodic structures can also be readily computed, provided that the extreme boundaries have simple conditions; otherwise, free evanescent wave motions are generated at the boundaries and the computation becomes more difficult. Periodic structure theory allows these computations to be undertaken by considering just one of the periodic elements of the whole system. The degrees of freedom which have to be considered are then very much fewer than those of the whole system, so the use of periodic structure theory can greatly increase the efficiency of calculation. The methods of finding natural frequencies of finite periodic structures with rectangular plate elements do not apply when the elements are non-rectangular (i.e., parallelograms) and further research is required into the wave motion in such structures.

Research into wave propagation in disordered structures is still in its relative infancy, and needs to be extended from one-dimensional mono-coupled systems through to two-dimensional multi-coupled systems. Particular attention should be given to comparing the forced responses of ordered and disordered periodic plates and shells. The maximum responses of disordered periodic systems can be expected to be higher than those of the ordered system in the vicinity of a localized source and over the whole structure excited by a distributed force.

Despite the numerous studies of wave motion in continuous periodic systems over the past 40 years, a simple physical explanation has yet to be presented for the very existence of frequency-propagation zones and attenuation zones. However, even if there is no simple answer to the question “Why does wave motion of one frequency propagate freely while motion of another frequency is attenuated?”, reliable prediction methods do exist for the properties of free motion and the magnitudes of forced motion.

REFERENCES

2. Lord Rayleigh 1887 Philosophical Magazine XXIV, 145–159. On the maintenance of vibrations by forces of double frequency, and on the propagation of waves through a medium endowed with a periodic structure.
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